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The rest points close to the strategies of these classes are asymptotically stable and all rest points other than these are not.

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The dynamical stability for an evolutionary language game under selection-mutation dynamics*

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October 22, 2017

Abstract

We present complete results pertaining to the dynamical stability for sender-receiver games following Lewis (1969), and Nowak and Krakauer (1999) under the selection-mutation dynamics. Our research reveals that two distinct classes of neutrally stable strategies have a distinguishing feature of the dynamic stability. The rest points close to the strategies of these classes are asymptotically stable and all rest points other than these are not.

JEL classification: C72; C73

Keywords: Language evolution; Sender-receiver game; Neutral stability; Selection-mutation dynamics

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1 Introduction

Lewis (1969) considered natural language as a convention in a community and analyzed it using the concept of Nash equilibrium. Faithfully following Lewis's idea, Wärneryd (1993) represents such communication between two players in a sender-receiver game. We also consider a sender-receiver game consisting of two players, a sender and a receiver. At the beginning of the game, the sender observes a state of the world picked up by Nature according to the uniform distribution on the set of states. After observing a state, the sender chooses a signal from a set of signals. Subsequently, the receiver is informed the signal chosen by the sender and associates the signal with a state in the set of states. If the state associated by the receiver coincides with the initial state observed by the sender, then each player gets a payoff of 1, and 0 otherwise.

Let's consider a situation where the set of states consists of three states and the set of signals also three signals.[1] This situation is formulated by a pair of 3×3 stochastic matrices, (P, Q), of which each entry is denoted by $p_{ij} \geq 0$ and $q_{ji} \geq 0$ respectively, that is,

$$(P, Q) = \left[\begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}, \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix} \right],$$

where $\sum_{j=1}^{3} p_{ij} = 1$ for each $i = 1, 2, 3$, and $\sum_{i=1}^{3} q_{ji} = 1$ for each $j = 1, 2, 3$. Each p_{ij} is the probability that the sender chooses a signal j after observing a state i, i.e., the sender's behavior strategy. Each q_{ji} is the probability that

[1] We shall formally give the model of more general cases in the next section.

the sender associates the given signal j with a state i, i.e., the receiver's behavior strategy.[2] Afterwards we call this situation the 3×3 one. Given such a (P, Q), the payoff for each player is equally given by

$$\frac{1}{3}\mathrm{tr}(PQ) := \frac{1}{3}\sum_{i=1}^{3}(\sum_{k=1}^{3} p_{ij}q_{ji}).$$

Since each signal does not have any pre-assigned meaning, we are interested in dynamically stable states of a population where many senders and many receivers are randomly and repeatedly matched to play the sender-receiver games described above. This is why we examine the symmetric equilibrium strategy of this sender-receiver games in the context of normal evolutionary game theory. Trapa and Nowak (2000) show that an evolutionary stable strategy of a sender-receiver game is a pair of a permutation matrix P and its transpose matrix Q. In the above 3×3 situation, a pair of

$$(P, Q) = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right]$$

is one of the class of the evolutionary stable strategies. Their finding implies that the sender-receiver games of each $n \times m$ situation with $n \neq m$ does not have the evolutionally stable strategy. From a static standpoint, Pawlowitsch (2008) rigorously characterizes the class of neutrally stable strategies in general situations where the number of possible states is not necessarily equal to that of available signals. While neutrally stable strategies are Lya-

[2]This formulation of a sender-receiver game follows the style of Nowak and Krakauer (1999).

punov stable under the replicator dynamics, Hofbauer and Hutteger (2015) investigate the dynamic stability of the rest point close to each neutrally stable strategy under the selection-mutation dynamics that is a perturbation of replicator dynamics with small mutation rates. They focus on the sender-receiver games of the 3×3 situations above and prove the existence of certain rest points of the dynamics close to Nash strategies of the sender-receiver games.[3] Furthermore, they show that all but the rest points of the dynamics close to the evolutionally stable strategies are dynamically unstable. More specifically, they find the first-order approximated rest points of the dynamics close to typical neutrally stable strategies as following,

$$\left[\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}\right], \left[\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ \nu & \nu & 1-2\nu \end{pmatrix}\right],$$

where $\nu < \frac{1}{2}$, and that each family of those rest points converges to the corresponding typical neutrally stable strategy above as each mutation rate of the dynamics goes to zero.[4] While there exist those rest points close to these typical neutrally stable strategies, they show that those rest points lose asymptotic stability under the selection-mutation dynamics. We follow their course in research and extend the object of investigation to the sender-receiver games with n states and m signals, that is, we admit that the number

[3] Hofbauer and Hutteger (2008) investigate binary sender-receiver games consisting of two states of the world and two signals, using the same selection-mutation dynamics. They indicate that the long-term behavior of the dynamics depends on the mutation rates.

[4] In each square bracket, the former parenthesis is the sender's P and the latter one the receiver's Q.

of possible states is *not* equal to that of available signals in our setup.

Although there is no evolutionally stable strategy in the sender-receiver games with $n \neq m$ (Trapa and Nowak, 2000), we find two distinct classes of neutrally stable strategies that each neighborhood of the strategy in these classes contains a rest point that can be asymptotically stable under the selection-mutation dynamics. The first class is the set of neutrally stable strategies that have a stretched form of the evolutionally stable strategy in a certain way.[5] We name the neutrally stable strategy in this class an *extended-signaling system*, following Lewis (1969) and Trapa and Nowak (2000), who called the evolutionary stable strategy in the sender-receiver games a *signaling sytem*.

The second clsss is a class of neutrally stable strategies with $|m - n| = 1$ and auxiliary conditions. We name the strategy in this class a *particular-hybrid system*. We show that each particular-hybrid system has an asymptotically stable rest point close to itself. At Last, we show that all Nash strategies but these two strategies are not asymptotically stable.

Hofbauer and Huttegger (2015) argues that "the second-order forces that are governed by mutation can increase the chance of successful signaling." Observing the dynamic stability of an exteded-signaling system and a particular-hybrid system in this note, their statement above is valid even if the number of states, n, is different from the number of signals, m.[6]

Finally, we present the properties of all the Nash strategies for the rest point near itself not to be asymptotically stable. We show that all Nash

[5]In Section 3 we shall propose the definition of these classes and precisely give the meaning of the word '*stretched*'.

[6]For the most part, we describe the case $n \leq m$ since the argument for another case $n > m$ is similar.

strategies but extended-signaling systems and particular hybrid systems are not asymptotically stable.

The rest of this note is organized as follows. In Section 2, we present the formal model of a sender-receiver game and the definition of the selection-mutation dynamics for our model. In Section 3, we propose our definition of an extended-signaling system and prove that a rest point close to each extended-signaling system exists. Moreover, we explicitly describe the characteristic polynominal of the first-order approximated Jacobian matrix at a rest point close to each extended-signaling system and show the conditions among parameters such as mutation rates, the number of states n, and that of signals m, for the rest point to be asymptotically stable. Subsequently, we propose our definition of a particular hybrid-sytem and determine whether a rest point close to a particular hybrid-system exists. Moreover, we explicitly describe the the characteristic polynomial of the first-order approximated Jacobian matrix at the rest point close to a particular hybrid-system and investigate the dynamic property of the rest point. In addition, we prove the rest point close to other systems to be unstable. Section 4 concludes the note.

2 The model

Our sender-receiver game consists of two players; one is a sender and the other is a receiver. Suppose that there are n states of the world (given by the set $N = \{1, 2, \ldots, n\}, n \geq 2$) and m signals (given by the set $M = \{1, 2, , \ldots, m\}, m \geq 2$). In this note, we explore the case $n \neq m$.

This game proceeds as follows. In the first stage, Nature chooses a state of the world $i \in N$ according to the uniform probability distribution of the set of states N, and the sender only learns the choice. In the second stage, the sender chooses a signal $j \in M$ according to her mixed strategy, represented by an $n \times m$ stochastic matrix P. Each entry p_{ij} of P is the probability that the sender chooses a signal $j \in M$ given a state $i \in N$ that Nature has chosen. [7] The set of the sender's mixed strategies is denoted by $P^{\triangle}_{n\times m}$. The receiver observes only the signal by the sender, not the choice of Nature. In the third stage, the receiver associates the observed signal $j \in M$ with a state $k \in N$ according to his mixed strategy, represented by an $m \times n$ stochactic matrix Q. Each entry q_{jk} of Q is the probability that the receiver associates the observed signal $j \in M$ with a state $k \in N$.[8] The set of the receiver's mixed strategies is denoted by $Q^{\triangle}_{m\times n}$. Finally, the payoff for each player is realized and the game ends. For each pair of strategies $(P, Q) \in P^{\triangle}_{n\times m} \times Q^{\triangle}_{m\times n}$, the payoff of each player is equally given by

$$\pi(P, Q) = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}q_{ji} = \frac{1}{n}tr(PQ).$$

Following the tradition of evolutionary game theory, we consider a population that is consisting of many players who play the sender-receiver game described above with a randomly matched opponent. We think a pair of a sender and a receiver matrix (P, Q) to be each player's strategy, and $P^{\triangle}_{n\times m} \times Q^{\triangle}_{m\times n}$ is interpreted not only as the set of mixed strategies but also the population's average strategy. In a match, we assume that each player

[7] Each ith row of P is $(p_{i1}, p_{i2}, \dots, p_{im})$ with $\sum_{j=1}^{m} p_{ij} = 1$.

[8] Each jth row of Q is $(q_{j1}, q_{i2}, \dots, q_{jn})$ with $\sum_{i=1}^{n} q_{ji} = 1$.

finds herself or himself in the role of the sender or the receiver with equal probabilities. In a match of a player with a strategy (P, Q) and the opponent with a strategy (P', Q'), the payoff for each player is given by

$$F[(P, Q), (P', Q')] = \frac{1}{2}\pi(PQ') + \frac{1}{2}\pi(P'Q).$$

We explore the game $\Gamma_{n,m} = \{P^{\triangle}_{n\times m} \times Q^{\triangle}_{m\times n}, F[(P, Q), (P', Q')]\}$ and its Nash equilibrium which is an equilibrium composition of the population.

Following a usual convention, a strategy played in a symmetric Nash equilibrium, i.e., a strategy that is a best response to itself, is called a Nash strategy. Pawlowitsch (2008) shows that a pair $(P, Q) \in P^{\triangle}_{n\times m} \times Q^{\triangle}_{m\times n}$ is a Nash strategy of $\Gamma_{n,m}$ if and only if $P \in B(Q)$ and $Q \in B(P)$, where $B(Q) \in P^{\triangle}_{n\times m}$ and $B(P) \in Q^{\triangle}_{m\times n}$ denote the best-response correspondence of the sender and receiver, respectively. $\Gamma_{n,m}$ has an abundance of Nash strategies, but might have no evolutionally stable strategy (ESS) because we allow the case $n \neq m$ (Trapa and Nowak, 2000). Hence, we turn to a weaker evolutionally stability concept than ESS to narrow down the set of Nash strategies.

Definition 1. A strategy $(P, Q) \in \mathcal{P}^{\triangle}_{n\times m} \times \mathcal{Q}^{\triangle}_{m\times n}$ is neutrally stable if
(i) (P, Q) is a Nash strategy and
(ii) whenever $(P', Q') \in B(Q) \times B(P)\backslash\{(P, Q)\}$, $\pi(P, Q) \geq \pi(P', Q')$.

Pawlowitsch (2008) characterizes of the neutrally stable strategies of $\Gamma_{n,m}$ with rigorous proofs.

Proposition 1. Let $(P, Q) \in \mathcal{P}^{\triangle}_{n\times m} \times \mathcal{Q}^{\triangle}_{m\times n}$ be a Nash strategy. (P, Q) is neutrally stable if and only if

(i) at least one of the two matrices, P or Q, has no zero column, and
(ii) neither P nor Q has a column with multiple maximal elements that are strictly between 0 and 1.

Although the set of neutrally stable strategies of $\Gamma_{n,m}$ turns out to be much smaller than the set of Nash strategies, $\Gamma_{n,m}$ has yet a large set of neutrally stable strategies. To select a certain strategy from among a set of neutrally stable strategies, we use an explicitly formulated dynamic selection process for strategy distribution with small noises.

Population Dynamics

We consider an $(m-1)$-dimensional simplex $S_i = \{(p_{i1}, p_{i2}, \ldots, p_{im}) | \sum_{j=1}^{m} p_{ij} = 1, p_{ij} \geq 0 \text{ for each } j \in M\}$ to be the set of population distributions over the set of pure strategies for senders who have observed a state $i \in N$ that Nature has chosen in the first stage. Similarly we consider an $(n-1)$-dimensional simplex $S_j = \{(q_{j1}, q_{j2}, \ldots, q_{jn}) | \sum_{k=1}^{n} q_{jk} = 1, q_{jk} \geq 0 \text{ for each } j \in N\}$ to be the set of population distributions of pure strategies for receivers who have observed a signal $j \in M$ that the opponent senders have chosen in the second stage. The set of population states is defined as $S = (\Pi_{i \in N} S_i) \times (\Pi_{j \in M} S_j)$, which stands for the set of behavioral strategies in the signaling game.

Our dynamic selection process is described by a dynamical system of differential equations defined for all points in S. The dynamical system is formulated as the following $2mn$ differential equations:

$$\dot{p}_{ij} = p_{ij}(q_{ji} - \sum_{s \in M}^{m} p_{is} q_{si}) + \varepsilon(1 - m p_{ij}),$$

$$\dot{q}_{ji} = q_{ji}(p_{ij} - \sum_{t \in N}^{n} q_{jt}p_{tj}) + \delta(1 - nq_{ji}),$$

where ε and δ are small, uniform mutation parameters. We denote this system by $S' = \Phi(S)$. This dynamical system is called the selection-mutation dynamics (Hofbauer, 1985). If $\varepsilon = \delta = 0$, the selection-mutation dynamics coincides with the replicator dynamics.

3 Results

We propose two category of equilibrium concepts that satisfy the neutrally stability defined above and some additional properties. We refer to the strategy in the first category as an *extended-signaling system* and the strategy in the second category as a *particular hybrid system*. We shall show that the strategy of the first category has its neighborhood that contains a rest point which can be asymptotically stable under our selection-mutation dynamics and that in a special case, $|m - n| = 1$, of our sender-receiver game, neighborhood of the strategy with the second category contains a rest point which can be asymptotically stable under the selectin-mutation dynamics. However, we find that a rest point close to other systems is not asymptotically stable.

3.1 Extended-signaling systems

We introduce some notations to simplify our expression of a extended-signaling system. We denote by $\mathbf{p}_j$ the jth column vector of the matrix P which is a strategy for the sender, and by $\mathbf{q}_i$ the ith column vector of the matrix

Q which is a strategy for the receiver, i.e., $P = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_m)$, and $Q = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n)$. Further, let $Z_P = \{j \in M \mid \mathbf{p}_j$ is a zero-column of the matrix $P\}$ and $Z_Q = \{i \in N \mid \mathbf{q}_i$ is a zero-column of the matrix $Q\}$.

Definition 2. We say that a strategy $(P^*, Q^*) \in \mathcal{P}^{\triangle}_{n\times m} \times \mathcal{Q}^{\triangle}_{m\times n}$ is an extended-signaling system if the following properties hold:

whenever $n \leq m$,

(i) $|Z_{P^*}| = m - n$,

(ii) $p^*_{ij} = q^*_{ji} = 1$ or $p^*_{ij} = q^*_{ji} = 0$ for each $i \in N$ and $j \in M \backslash Z_P$,

(iii) $q^*_{ji} = \frac{1}{n}$ for each $j \in Z_P$ and $i \in N$;

whenever $n > m$,

(i) $|Z_{Q^*}| = n - m$,

(ii) $p^*_{ij} = q^*_{ji} = 1$ or $p^*_{ij} = q^*_{ji} = 0$ for each $j \in M$ and $i \in N \backslash Z_Q$,

(iii) $p^*_{ij} = \frac{1}{m}$ for each $i \in Z_Q$ and $j \in M$;

where $\left(p^*_{ij}, q^*_{ji}\right)_{(i,j)\in N\times M}$ denote the entries of the extended-signaling system (P^*, Q^*).

The strategies in Example 1 below, (P^*_1, Q^*_1) and (P^*_2, Q^*_2), are those of extended-signaling systems.[9] From Proposition 1, we can see that not only these examples but the extended-signaling system generally has neutral stability.

Example 1.

$$P^*_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q^*_1 = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix},$$

[9] Here, ϕ denotes an empty set.

where $Z_{P_1^*} = \{2\}, Z_{Q_1^*} = \phi$.

$$P_2^* = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 \end{pmatrix}, Q_2^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

where $Z_{P_2^*} = \phi, Z_{Q_2^*} = \{2, 4\}$.

We introduce additional notatins to give clear expositions in this subsections. Given an extended-signaling system (P^*, Q^*), we divide the set $N \times M$ into subsets, $I_i, i = 1, 2, 3$ for the case $n \leq m$, and $I_i', i = 1, 2, 3$ for the case $n > m$:

$I_1 = \{(i, j) \in N \times M \mid p_{ij}^* = 1\}$,

$I_2 = \{(i, j) \in N \times M \mid j \notin Z_{P^*}, p_{ij}^* = 0\}$,

$I_3 = \{(i, j) \in N \times M \mid j \in Z_{P^*}\}$;

$I_1' = \{(i, j) \in N \times M \mid q_{ij}^* = 1\}$,

$I_2' = \{(i, j) \in N \times M \mid i \notin Z_{Q^*}, q_{ji}^* = 0\}$,

$I_3' = \{(i, j) \in N \times M \mid i \in Z_{Q^*}\}$.

For (P_1^*, Q_1^*) in Example 1, we can easily check that $I_1 = \{(1, 1), (2, 3)\}, I_2 = \{(1, 3), (2, 1)\}, I_3 = \{(1, 2), (2, 2)\}$, and for (P_2^*, Q_2^*), $I_1' = \{(1, 1), (3, 3), (5, 2)\}, I_2' = \{(1, 2), (1, 3), (3, 1), (3, 2), (5, 1), (5, 3)\}, I_3' = \{(2, 1), (2, 2), (2, 3), (4, 1), (4, 2), (4, 3)\}$.

We find a rest point of the selection-mutation dynamics close to each extended-signaling system. A rest point of our dynamical system, $S' = \Phi(S)$, is generally defined as a point that satisfies $\Phi\big((p_{ij}, q_{ji})_{(i,j) \in N \times M}\big) = \mathbf{0}$, where $\mathbf{0}$ is a zero-column vector. Especially, we focus on a rest point called to be symmetric for the corresponding extended-signaling system.

Definition 3. Let (P^*, Q^*) be an extended-signaling system. We say that a rest point $(\tilde{P}(\varepsilon,\delta), \tilde{Q}(\varepsilon)) \in \mathbf{S}$ has a symmetric form for $(P(\varepsilon,\delta), Q(\varepsilon,\delta))$ if

· For $n \le m$, for some real values $\varepsilon_1, \varepsilon_2, \delta_1, \delta_2$,

$$\tilde{p}_{ij}(\varepsilon,\delta) = \begin{cases} 1-(n-1)\varepsilon_1-(m-n)\varepsilon_2 & \text{for each } (i,j) \in I_1, \\ \varepsilon_1 & \text{for each } (i,j) \in I_2, \\ \varepsilon_2 & \text{for each } (i,j) \in I_3; \end{cases}$$

$$\tilde{q}_{ji}(\varepsilon,\delta) = \begin{cases} 1-(n-1)\delta_1 & \text{for each } (i,j) \in I_1, \\ \delta_1 & \text{for each } (i,j) \in I_2, \\ q_1 & \text{for each } (i,j) \in I_3, \end{cases}$$

where $\big(\tilde{p}_{ij}(\varepsilon,\delta), \tilde{q}_{ji}(\varepsilon,\delta)\big)_{(i,j)\in N\times M}$ are entries of $(\tilde{P}(\varepsilon,\delta), \tilde{Q}(\varepsilon,\delta))$.

Example 2.

For an extended-signaling system (P_3^*, Q_3^*),

$$P_3^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad Q_3^* = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 \end{pmatrix}.$$

the symmetric rest point of the selection-mutation dynamics has a form such that

$$\tilde{P}_3(\varepsilon,\delta) = \begin{pmatrix} 1-2\varepsilon_1-2\varepsilon_2 & \varepsilon_2 & \varepsilon_1 & \varepsilon_2 & \varepsilon_1 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_1 & \varepsilon_2 & 1-2\varepsilon_1-2\varepsilon_2 \\ \varepsilon_1 & \varepsilon_2 & 1-2\varepsilon_1-2\varepsilon_2 & \varepsilon_2 & \varepsilon_1 \end{pmatrix},$$

$$\tilde{Q}_3(\varepsilon,\delta) = \begin{pmatrix} 1-2\delta_1 & \delta_1 & \delta_1 \\ q_1 & q_1 & q_1 \\ \delta_1 & \delta_1 & 1-2\delta_1 \\ q_1 & q_1 & q_1 \\ \delta_1 & 1-2\delta_1 & \delta_1 \end{pmatrix},$$

where $Z_{P^*} = \{2,4\}, I_1 = \{(1,1),(3,3),(2,5)\}$,

$I_2 = \{(2,1),(3,1),(1,3),(2,3),(1,5),(3,5)\}$,

$I_3 = \{(1,2),(2,2),(3,2),(1,4),(2,4),(3,4)\}$.

Theorem 1. Let $(P^*,Q^*) \in \mathcal{P}^{\triangle}_{n\times m} \times \mathcal{Q}^{\triangle}_{m\times n}$ be an extended-signaling system. Then, for each pair of mutation rates, (ε,δ), there exists a neighborhood of the point (P^*,Q^*) that contains a unique symmetric rest point, $(\tilde{P}(\varepsilon,\delta),\tilde{Q}(\varepsilon,\delta))$, of the selection-mutation dynamics.

Proof. We consider the case $n \leq m$ and abbreviate the proof for the case $n > m$ because it is similar to that of the former.

Fix a signal j of $(i,j) \in I_3$ to be $\bar{j}$. Since $\sum_{i=1}^{n} q_{\bar{j}i} = nq_1 = 1$, we have $q_1 = \frac{1}{n}$.

We sequentially find the values of the entries of the symmetric rest point, $\varepsilon_1, \varepsilon_2$, and δ_1 consistent with the conditions required for the rest point, $\dot{p} = 0$ and $\dot{q} = 0$.

We write down our dynamical system $S' = \Phi(S)$ of the selection-mutation dynamics:

$$
\begin{cases}
\dot{p}_{11} & = & p_{11}(q_{11} - p_{11}q_{11} - p_{12}q_{21} - \cdots - p_{1m}q_{m1}) + \varepsilon(1 - mp_{11}), \\
\dot{p}_{12} & = & p_{12}(q_{21} - p_{11}q_{11} - p_{12}q_{21} - \cdots - p_{1m}q_{m1}) + \varepsilon(1 - mp_{12}), \\
 & \vdots & \\
\dot{p}_{nm} & = & p_{nm}(q_{mn} - p_{n1}q_{1n} - p_{n2}q_{2n} - \cdots - p_{nm}q_{mn}) + \varepsilon(1 - mp_{nm}), \\
\dot{q}_{11} & = & q_{11}(p_{11} - q_{11}p_{11} - q_{12}p_{21} - \cdots - q_{1n}p_{n1}) + \delta(1 - nq_{11}), \\
\dot{q_{12}} & = & q_{12}(p_{21} - q_{11}p_{11} - q_{12}p_{21} - \cdots - q_{1n}p_{n1}) + \delta(1 - nq_{12}), \\
 & \vdots & \\
\dot{q}_{mn} & = & q_{mn}(p_{nm} - q_{m1}p_{1m} - q_{m2}p_{2m} - \cdots - q_{mn}p_{nm}) + \delta(1 - nq_{mn}).
\end{cases}
$$

By removing equations, $\dot{p}_{ij} = 0$ and $\dot{q}_{ij} = 0$ with $(i,j) \in I_1$, and $\dot{q}_{ij} = 0$ with $(i,j) \in I_3$, from our whole system, $\Phi(S) = \mathbf{0}$, we obtain a following sub-system of $\Phi(S) = \mathbf{0}$, which is denoted by $F = \mathbf{0}$.

$$
F = \mathbf{0} \Leftrightarrow \begin{cases}
\dot{p}_{ij} = 0 \ \text{ for each } (i,j) \in I_2, \\
\dot{p}_{ij} = 0 \ \text{ for each } (i,j) \in I_3, \\
\dot{q}_{ji} = 0 \ \text{ for each } (i,j) \in I_2.
\end{cases} .
$$

By substituting the entries $(\tilde{p}_{ij}, \tilde{q}_{ji})$ of Definition 3 and $q_1 = \frac{1}{n}$ into the sub-system $F = \mathbf{0}$, we get a following reduced system, $f(\varepsilon_1, \varepsilon_2, \delta_1; \varepsilon, \delta) = \mathbf{0}$, consisting only of three equations, $f_i(\varepsilon_1, \varepsilon_2, \delta_1; \varepsilon, \delta) = 0, i = 1, 2, 3.$

$$
F(\varepsilon_1, \varepsilon_2, \delta_1; \varepsilon, \delta) = \mathbf{0} \Leftrightarrow \begin{cases}
\tilde{p}_{ij}[\tilde{q}_{ij} - \displaystyle\sum_{(i,s)\in N\times M} \tilde{p}_{is}\tilde{q}_{sj}] + \varepsilon(1 - m\tilde{p}_{ij}) = 0 \ \text{ for each } (i,j) \in I_2, \\
\tilde{p}_{ij}[\tilde{q}_{ij} - \displaystyle\sum_{(i,s)\in N\times M} \tilde{p}_{is}\tilde{q}_{sj}] + \varepsilon(1 - m\tilde{p}_{ij}) = 0 \ \text{ for each } (i,j) \in I_3, \\
\tilde{q}_{ji}[\tilde{p}_{ij} - \displaystyle\sum_{(t,j)\in N\times M} \tilde{q}_{jt}\tilde{p}_{tj}] + \delta(1 - n\tilde{q}_{ji}) = 0 \ \text{ for each } (i,j) \in I_2,
\end{cases} .
$$

$$\Leftrightarrow \begin{cases} \tilde{p}_{ij}[\tilde{q}_{ij} - \sum_{(i,s)\in I_1} \tilde{p}_{is}\tilde{q}_{sj} - \sum_{(i,s)\in I_2} \tilde{p}_{is}\tilde{q}_{sj} - \sum_{(i,s)\in I_3} \tilde{p}_{is}\tilde{q}_{sj}] + \varepsilon(1 - m\tilde{p}_{ij}) = 0 \text{ for each } (i,j) \in I_2, \\ \tilde{p}_{ij}[\tilde{q}_{ij} - \sum_{(i,s)\in I_1} \tilde{p}_{is}\tilde{q}_{sj} - \sum_{(i,s)\in I_2} \tilde{p}_{is}\tilde{q}_{sj} - \sum_{(i,s)\in I_3} \tilde{p}_{is}\tilde{q}_{sj}] + \varepsilon(1 - m\tilde{p}_{ij}) = 0 \text{ for each } (i,j) \in I_3, \\ \tilde{q}_{ji}[\tilde{p}_{ji} - \sum_{(t,j)\in I_1} \tilde{q}_{jt}\tilde{p}_{tj} - \sum_{(t,j)\in I_2} \tilde{q}_{jt}\tilde{p}_{tj} - \sum_{(t,j)\in I_3} \tilde{q}_{jt}\tilde{p}_{tj}] + \delta(1 - n\tilde{q}_{ji}) = 0 \text{ for each } (i,j) \in I_2, \end{cases}$$

$$\Leftrightarrow \begin{cases} \varepsilon_1 \left\{\delta_1 - (n-1)\varepsilon_1\delta_1 - [1-(n-1)\varepsilon_1 - (m-n)\varepsilon_2]\,[1-(n-1)\delta_1] - \frac{1}{n}(m-n)\varepsilon_2\right\} + \varepsilon(1-m\varepsilon_1) = 0, \\ \varepsilon_2 \left\{\frac{1}{n} - (n-1)\varepsilon_1\delta_1 - [1-(n-1)\varepsilon_1 - (m-n)\varepsilon_2]\,[1-(n-1)\delta_1] - \frac{1}{n}(m-n)\varepsilon_2\right\} + \varepsilon(1-m\varepsilon_2) = 0, \\ \delta_1 \left\{\varepsilon_1 - (n-1)\delta_1\varepsilon_1 - [1-(n-1)\delta_1]\,[1-(n-1)\varepsilon_1 - (m-n)\varepsilon_2]\right\} + \delta(1-n\delta_1) = 0. \end{cases}$$

From this, we see that $(\varepsilon_1, \varepsilon_2, \delta_1; \varepsilon, \delta) = (0,0,0;0,0)$ is a solution to the reduced system, i.e., $f_i(0,0,0;0,0) = 0, i = 1,2,3$.

Let Df denote the Jacobian matrix of f_1, f_2, f_3 with respect to $\varepsilon_1, \varepsilon_2, \delta_1$, that is,

$$Df = \begin{pmatrix} \frac{\partial f_1}{\partial \varepsilon_1} & \frac{\partial f_1}{\partial \varepsilon_2} & \frac{\partial f_1}{\partial \delta_1} \\ \frac{\partial f_2}{\partial \varepsilon_1} & \frac{\partial f_2}{\partial \varepsilon_2} & \frac{\partial f_2}{\partial \delta_1} \\ \frac{\partial f_3}{\partial \varepsilon_1} & \frac{\partial f_3}{\partial \varepsilon_2} & \frac{\partial f_3}{\partial \delta_1} \end{pmatrix}.$$

At the point $(\varepsilon_1, \varepsilon_2, \delta_1; \varepsilon, \delta) = (0,0,0;0,0)$, we have

$$\det(Df(\mathbf{0})) = \begin{vmatrix} -1 & 0 & 0 \\ 0 & \frac{1-n}{n} & 0 \\ 0 & 0 & -1 \end{vmatrix} = \frac{1-n}{n} \neq 0,$$

where $\det(Df(\mathbf{x}))$ denotes the determinant of $Df(\mathbf{x})$ at point $\mathbf{x} = (\varepsilon_1, \varepsilon_2, \delta_1; \varepsilon, \delta)$. The last inequality holds because we assume $n \geq 2$ in our model. By the implicit function theorem, our reduced system, $f_i(\varepsilon_1, \varepsilon_2, \delta_1; \varepsilon, \delta) = 0, i = 1,2,3$, defines $\varepsilon_1, \varepsilon_2$, and δ_1 as continuously differentiable functions of ε and δ in some neighborhood of $(\varepsilon_1, \varepsilon_2, \delta_1; \varepsilon, \delta) = (0, 0, 0; 0, 0)$.[10] We denote these functions by $\varepsilon_1(\varepsilon, \delta), \varepsilon_2(\varepsilon, \delta)$, and $\delta_1(\varepsilon, \delta)$.

Inserting $q_{ji} = \frac{1}{n}$ and $p_{ij} = \varepsilon_2$ into the following equations which have been removed from our whole sytem, $\Phi(S) = \mathbf{0}$, to construct the previous sub-system, $f_i(\varepsilon_1, \varepsilon_2, \delta_1; \varepsilon, \delta) = 0, i = 1, 2, 3$.

$$\dot{q}_{ji} = q_{ji}(p_{ij} - \sum_{t=1}^{n} q_{jt}p_{tj}) + \delta(1 - nq_{ji}) \quad \text{with } (i, j) \in I_3,$$

we have

$$\dot{q}_{ji} = \tilde{q}_{ji}(\tilde{p}_{ij} - \sum_{t=1}^{n} \tilde{q}_{jt}\tilde{p}_{tj}) + \delta(1 - n\tilde{q}_{ji}) = \frac{1}{n}(\varepsilon_2 - n \times \frac{1}{n} \times \varepsilon_2) + \delta(1 - n \times \frac{1}{n}) = 0.$$

The constant values, $\tilde{q}_{ji} = \frac{1}{n}$ and $\tilde{p}_{ij} = \varepsilon_2$ for each $(i, j) \in I_3$, of the symmetric rest point are consistent with the condition required of the rest point, $\dot{q}_{ji} = 0$.

It remains to prove that the functions $\varepsilon_1(\varepsilon, \delta), \varepsilon_2(\varepsilon, \delta)$, and $\delta_1(\varepsilon, \delta)$ satisfy the equations $\dot{p}_{ij} = 0$ and $\dot{q}_{ji} = 0$ with $(i, j) \in I_1$ which have been removed. From our fixed form of the rest point, we can assert that for each $(i, j) \in I_2$, $\dot{p}_{ij} = \dot{\varepsilon_1}(\varepsilon, \delta) = 0$, $\dot{q}_{ji} = \dot{\delta}_1(\varepsilon, \delta) = 0$, and for each $(i, j) \in I_3$, $\dot{p}_{ij} = \dot{\varepsilon_2}(\varepsilon, \delta) = 0$. Since we have also fixed $p_{ij} = 1-(n-1)\varepsilon_1-(m-n)\varepsilon_2$ and $q_{ji} = 1-(n-1)\delta_1$ for each $(i, j) \in I_1$, we get $\dot{p}_{ij} = -(n-1)\dot{\varepsilon_1}(\varepsilon, \delta) - (m-n)\dot{\varepsilon_2}(\varepsilon, \delta) = 0$ and $\dot{q}_{ji} = -(n-1)\dot{\delta}_1(\varepsilon, \delta) = 0$.

Therefore, we conclude that for all $(i, j) \in M \times N$, $\dot{p}_{ij} = \dot{q}_{ji} = 0$ with $(\varepsilon_1, \varepsilon_2, \delta_1) = (\varepsilon_1(\varepsilon, \delta), \varepsilon_2(\varepsilon, \delta), \delta_1(\varepsilon, \delta))$, which proves the theorem. □

[10] That is, we get a unique candidate of values of $\varepsilon_1, \varepsilon_2$, and δ_1, for each pair of mutation rates, (ε, δ).

We show the form of the first-order approximation to the symmetric rest point close to the corresponding exteded-signaling systems. From this form, we can easily see that an extended-signaling system is the limit point of the family of the symmetric rest points as (ε, δ) goes to $(0, 0)$. Furthermore, using this form, we explore the stability of the rest point of our dynamical system.

Corollary 1. The first-order approximated entries $\big(\tilde{p}_{ij}(\varepsilon, \delta), \tilde{q}_{ji}(\varepsilon, \delta)\big)_{(i,j)\in N\times M}$ of the rest point $(\tilde{P}(\varepsilon, \delta), \tilde{Q}(\varepsilon, \delta)) \in S$ in the neighborhood of an extended-signaling system (P^*, Q^*) is explicitly given as shown below: Moreover, $\lim\limits_{(\varepsilon,\delta)\to(0,0)} \big(\tilde{P}(\varepsilon, \delta), \tilde{Q}(\varepsilon, \delta)\big) = (P^*, Q^*)$.

· For $n \leq m$,

$$\tilde{p}_{ij}(\varepsilon, \delta) = \begin{cases} 1 + \frac{2n-mn-1}{n-1}\varepsilon & \text{for each } (i, j) \in I_1, \\ \varepsilon & \text{for each } (i, j) \in I_2, \\ \frac{n}{n-1}\varepsilon & \text{for each } (i, j) \in I_3; \end{cases}$$

$$\tilde{q}_{ji}(\varepsilon, \delta) = \begin{cases} 1 - (n-1)\delta & \text{for each } (i, j) \in I_1, \\ \delta & \text{for each } (i, j) \in I_2, \\ \frac{1}{n} & \text{for each } (i, j) \in I_3; \end{cases}$$

$\cdot$ For $n > m$,

$$\tilde{p}_{ij}(\varepsilon, \delta) = \begin{cases} 1 - (m-1)\varepsilon & \text{for each } (i,j) \in I_1', \\ \varepsilon & \text{for each } (i,j) \in I_2', \\ \frac{1}{m} & \text{for each } (i,j) \in I_3'; \end{cases}$$

$$\tilde{q}_{ji}(\varepsilon, \delta) = \begin{cases} 1 + \frac{2m-mn-1}{n-1}\delta & \text{for each } (i,j) \in I_1' \\ \delta & \text{for each } (i,j) \in I_2' \\ \frac{m}{m-1}\delta & \text{for each } (i,j) \in I_3', \end{cases}$$

Proof. We consider the case $n \leq m$ and abbreviate the proof for the case $n > m$ because it is similar to that of the former.

Tayler's formula for the function $(\varepsilon_1(\varepsilon, \delta), \varepsilon_2(\varepsilon, \delta), \delta_1(\varepsilon, \delta))$ about $(\varepsilon, \delta) = (0, 0)$ is given by

$$\begin{pmatrix} \varepsilon_1(\varepsilon, \delta) \\ \varepsilon_2(\varepsilon, \delta) \\ \delta_1(\varepsilon, \delta) \end{pmatrix} = \begin{pmatrix} \varepsilon_1(0,0) \\ \varepsilon_2(0,0) \\ \delta_1(0,0) \end{pmatrix} + \begin{pmatrix} \frac{\partial \varepsilon_1}{\partial \varepsilon}(0,0) & \frac{\partial \varepsilon_1}{\partial \delta}(0,0) \\ \frac{\partial \varepsilon_2}{\partial \varepsilon}(0,0) & \frac{\partial \varepsilon_2}{\partial \delta}(0,0) \\ \frac{\partial \delta_1}{\partial \varepsilon}(0,0) & \frac{\partial \delta_1}{\partial \delta}(0,0) \end{pmatrix} \begin{pmatrix} \varepsilon \\ \delta \end{pmatrix} + \begin{pmatrix} o_1(\varepsilon, \delta) \\ o_2(\varepsilon, \delta) \\ o_3(\varepsilon, \delta) \end{pmatrix}.$$

Because $(\varepsilon_1(0,0), \varepsilon_2(0,0), \delta_1(0,0))$ is a solution of the system,
$f_i(\varepsilon_1(0,0), \varepsilon_2(0,0), \delta_1(0,0); 0, 0) = 0, i = 1, 2, 3,$
we have $(\varepsilon_1(0,0), \varepsilon_2(0,0), \delta_1(0,0)) = (0,0,0)$.

By the implicit function theorem, we have

$$\begin{pmatrix} \frac{\partial \varepsilon_1}{\partial \varepsilon}(0,0) & \frac{\partial \varepsilon_1}{\partial \delta}(0,0) \\ \frac{\partial \varepsilon_2}{\partial \varepsilon}(0,0) & \frac{\partial \varepsilon_2}{\partial \delta}(0,0) \\ \frac{\partial \delta_1}{\partial \varepsilon}(0,0) & \frac{\partial \delta_1}{\partial \delta}(0,0) \end{pmatrix} = -(Df(\mathbf{0})^{-1}) \begin{pmatrix} \frac{\partial f_1}{\partial \varepsilon}(\mathbf{0}) & \frac{\partial f_1}{\partial \delta}(\mathbf{0}) \\ \frac{\partial f_2}{\partial \varepsilon}(\mathbf{0}) & \frac{\partial f_2}{\partial \delta}(\mathbf{0}) \\ \frac{\partial f_3}{\partial \varepsilon}(\mathbf{0}) & \frac{\partial f_3}{\partial \delta}(\mathbf{0}) \end{pmatrix}$$

$$= - \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1-n}{n} & 0 \\ 0 & 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{n}{n-1} & 0 \\ 0 & 1 \end{pmatrix},$$

where $\mathbf{0} = (0,0,0;0,0)$.

Tayler's formula descrived above becomes

$$\begin{pmatrix} \varepsilon_1(\varepsilon,\delta) \\ \varepsilon_2(\varepsilon,\delta) \\ \delta_1(\varepsilon,\delta) \end{pmatrix} = \begin{pmatrix} \varepsilon_1(0,0) \\ \varepsilon_2(0,0) \\ \delta_1(0,0) \end{pmatrix} + \begin{pmatrix} \frac{\partial \varepsilon_1}{\partial \varepsilon}(0,0) & \frac{\partial \varepsilon_1}{\partial \delta}(0,0) \\ \frac{\partial \varepsilon_2}{\partial \varepsilon}(0,0) & \frac{\partial \varepsilon_2}{\partial \delta}(0,0) \\ \frac{\partial \delta_1}{\partial \varepsilon}(0,0) & \frac{\partial \delta_1}{\partial \delta}(0,0) \end{pmatrix} \begin{pmatrix} \varepsilon \\ \delta \end{pmatrix} + \begin{pmatrix} o_1(\varepsilon,\delta) \\ o_2(\varepsilon,\delta) \\ o_3(\varepsilon,\delta) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ \frac{n}{n-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varepsilon \\ \delta \end{pmatrix} + \begin{pmatrix} o_1(\varepsilon,\delta) \\ o_2(\varepsilon,\delta) \\ o_3(\varepsilon,\delta) \end{pmatrix},$$

where $o_i(\varepsilon,\delta)$,$i = 1,2,3$ stands for the second- or higher-order terms of ε, and δ. Thus, we get the first-order approximated values of $\varepsilon_1, \varepsilon_2$, δ_1 respectively as follows:

$$\begin{aligned} \varepsilon_1 &= \varepsilon + o_1(\varepsilon,\delta), \\ \varepsilon_2 &= \tfrac{n}{n-1}\varepsilon + o_2(\varepsilon,\delta), \\ \delta_1 &= \delta + o_3(\varepsilon,\delta). \end{aligned}$$

Replacing ε_1, ε_2, and δ_1 in $\tilde{p}_{ij}(\varepsilon,\delta)$ and $\tilde{q}_{ij}(\varepsilon,\delta)$ for each $(i,j) \in M \times N$ of the form of the symmetric rest point, by these values, we find the first-order approximated rest point. $\square$

Example 3.

The first order approximation of the symmetric rest point close to the extended-signaling system (P_i^*, Q_i^*), $i = 1, 2$, in Example 1, is given by

$$\tilde{P}_1(\varepsilon,\delta) = \begin{pmatrix} 1-3\varepsilon & 2\varepsilon & \varepsilon \\ \varepsilon & 2\varepsilon & 1-3\varepsilon \end{pmatrix}, \tilde{Q}_1(\varepsilon,\delta) = \begin{pmatrix} 1-\delta & \delta \\ \frac{1}{2} & \frac{1}{2} \\ \delta & 1-\delta \end{pmatrix},$$

where $Z_{P_1^*} = \{2\}$.

$$\tilde{P}_2(\varepsilon,\delta) = \begin{pmatrix} 1-2\varepsilon & \varepsilon & \varepsilon \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \varepsilon & \varepsilon & 1-2\varepsilon \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \varepsilon & 1-2\varepsilon & \varepsilon \end{pmatrix}, \tilde{Q}_2(\varepsilon,\delta) = \begin{pmatrix} 1-5\delta & \frac{3}{2}\delta & \delta & \frac{3}{2}\delta & \delta \\ \delta & \frac{3}{2}\delta & \delta & \frac{3}{2}\delta & 1-5\delta \\ \delta & \frac{3}{2}\delta & 1-5\delta & \frac{3}{2}\delta & \delta \end{pmatrix},$$

where $Z_{Q_2^*} = \{2, 4\}$.

We sequentially demonstrate how the asymptotic stability of the selection-mutation dynamics at the symmetric rest point depends on the rates of mutations, ε and δ, as well as the numbers of states, n, and signals, m. Table 1 is an example of the first order approximated Jacobian matrix of our dynamical system evaluated at the symmetric rest point close to an exteded-signaling system (P_4^*, Q_4^*) for $n = 2$ and $m = 3$. Each entry of this table is $\frac{[\partial \text{ the row of this table}]}{[\partial \text{ the column of this table}]}$ e.g., $\frac{\partial p_{ij}}{\partial p_{st}}$. Observing this table, we expect that the symmetric rest point can be asymptotically stable.

We turn to the general case. Let $J\Phi(\varepsilon, \delta, n, m)$ denote the Jacobian matrix with respect to $\dot{p}_{ij}, \dot{q}_{ji}$ $(i = 1, \ldots, n, j = 1, \ldots, m)$ of our dynamical system with the mutation rate ε, δ, which is evaluated at the first-order approximated rest point near an extended-signaling system of our sender-receiver game with n states of the world and m signals. Table 2.A lists of *all* entries of $J\Phi(\varepsilon, \delta, n, m)$ for $n \leq m$.[11] $J\Phi(\varepsilon, \delta, n, m)$ is a square matrix with $(2nm)^2$ entries. This list consists of six sub-lists (rows) of entries of $J\Phi(\varepsilon, \delta, n, m)$ as follows:

1. $\frac{\partial \dot{p}_{ij}}{\partial p_\cdot}$ and $\frac{\partial \dot{p}_{ij}}{\partial q_\cdot}$ with $(i, j) \in I_1$,
2. $\frac{\partial \dot{p}_{ij}}{\partial p_\cdot}$ and $\frac{\partial \dot{p}_{ij}}{\partial q_\cdot}$ with $(i, j) \in I_2$,
3. $\frac{\partial \dot{p}_{ij}}{\partial p_\cdot}$ and $\frac{\partial \dot{p}_{ij}}{\partial q_\cdot}$ with $(i, j) \in I_3$,
4. $\frac{\partial \dot{q}_{ji}}{\partial p_\cdot}$ and $\frac{\partial \dot{q}_{ji}}{\partial q_\cdot}$ with $(i, j) \in I_1$,
5. $\frac{\partial \dot{q}_{ji}}{\partial p_\cdot}$ and $\frac{\partial \dot{q}_{ji}}{\partial q_\cdot}$ with $(i, j) \in I_2$,
6. $\frac{\partial \dot{q}_{ji}}{\partial p_\cdot}$ and $\frac{\partial \dot{q}_{ji}}{\partial q_\cdot}$ with $(i, j) \in I_3$.

Contents of each cell of sub-lists are explained in the following table.

	specifying the pair of (s, t)
specifying the pair of (i, j)	value of $\left(\frac{\partial \dot{p}_{ij}}{\partial p_{st}} \text{ or } \frac{\partial \dot{p}_{ji}}{\partial q_{ts}} \text{ or } \frac{\partial \dot{q}_{ji}}{\partial p_{st}} \text{ or } \frac{\partial \dot{q}_{ji}}{\partial q_{ts}}\right)$
number of such entries in the Jacobian matrix	[number of rows with the $(i, j) \in I_i (i = 1, 2, 3)$] $\times$ [number of entries in each row with $(s, t) \in I_i (i = 1, 2, 3)$]

Now using this list leads to the main result of our note.[12]

Theorem 2. If $n \leq m$ and $\frac{\varepsilon}{\delta} < \frac{n(n-1)}{mn-n^2-1}$ for sufficiently small ε and δ, then the symmetric rest point close to an extended-signaling system is asymptotically stable.

Proof. The characteristic equation of the first-order approximated Jacobian matrix evaluated at the symmetric rest point close to *any* extended-signaling

[11] Table 2.B lists the entries for $n \geq m$.

[12] We abbreviate the result for $n > m$, since it is similar to the case $n \leq m$.

system is given by

$$
\begin{aligned}
&[\lambda - \tfrac{mn-3n+2}{n-1}\varepsilon - (n-1)\delta + 1]^{n-1} \\
&\times[\lambda + \varepsilon - n\delta + 1]^{n(n-1)} \\
&\times[\lambda + \tfrac{n}{n-1}\varepsilon - (n-1)\delta + 1 - \tfrac{1}{n}]^{n(m-n)-1} \\
&\times[\lambda - \tfrac{-2n+mn+1}{n-1}\varepsilon - (n-2)\delta + 1]^{n} \\
&\times[\lambda - \tfrac{mn-n}{n-1}\varepsilon + \delta + 1]^{n(n-1)} \\
&\times[\lambda + \tfrac{1}{n-1}\varepsilon + n\delta]^{n(m-n)-1} \\
&\times[\lambda - \tfrac{mn-n^2-1}{n-1}\varepsilon + n\delta] \\
&\times[\lambda + \tfrac{1}{n-1}(n^3 - n^2 + n - mn^2 + mn)\varepsilon + (1-n)\delta + 1 - \tfrac{1}{n}] \\
&\times[\lambda + (mn - n^2 - n + 2)\varepsilon + (1-n)\delta + 1] = 0
\end{aligned}
$$

where λ is the eigenvalue.

We explain briefly the procedure for getting the above equation. Let $A = J\Phi - \lambda I$, that is, det$A$ denotes the characteristic polynomial. Let $a_{ij} \in A$ be the entries of the matrix A and A_{ij} the corresponding (i, j)th cofactor. In the following we disregard any term that is second- or higher-order one in ε, δ because of the continuity of the characteristic polynomial with respect to ε and δ.[13] Consequently, we may regard most of entries of the Jacobian matrix except its diagonal factors as 0 or linear forms of ε and δ. Referring to Table2.A and noting that $|I_1| = n, |I_2| = (n-1)$, and $|I_3| = n(m-n)$, we expand the matrix A along any ith row. We get the following polynomial.

[13]This is a normal procedure for determining the stability of a rest point, which Hofbouer and Hutteger (2015) also follows.

$$
\begin{aligned}
& \det A \\
= \ & \textstyle\sum_{k=1}^{2nm}(-1)^{i+k}a_{ik}A_{ik} \\
= \ & [-\lambda+\tfrac{mn-3n+2}{n-1}\varepsilon+(n-1)\delta-1]^{n}[-\lambda-\varepsilon+n\delta-1]^{n(n-1)}[\lambda-\tfrac{n}{n-1}\varepsilon+(n-1)\delta-1+\tfrac{1}{n}]^{n(m-n)} \\
& \times[-\lambda+\tfrac{-2n+mn+1}{n-1}\varepsilon+(n-2)\delta-1]^{n}[-\lambda+\tfrac{mn-n}{n-1}\varepsilon-\delta-1]^{n(n-1)}[-\lambda-\tfrac{1}{n-1}\varepsilon-n\delta]^{n(m-n)} \\
& -(-\tfrac{n}{n-1}\varepsilon)(-\tfrac{1}{n}-\tfrac{2n-mn-1}{n(n-1)}\varepsilon)[-\lambda+\tfrac{mn-3n+2}{n-1}\varepsilon+(n-1)\delta+1]^{n-1}[-\lambda-\varepsilon+n\delta+1]^{n(n-1)}[-\lambda-\tfrac{n}{n-1}\varepsilon+(n-1)\delta+1+\tfrac{1}{n}]^{n(m-n)-1} \\
& \times[-\lambda+\tfrac{-2n+mn+1}{n-1}\varepsilon+(n-2)\delta-1]^{n}[-\lambda+\tfrac{mn-n}{n-1}\varepsilon-\delta+1]^{n(n-1)}[-\lambda-\tfrac{1}{n-1}\varepsilon-n\delta]^{n(m-n)}\times n(m-n) \\
& -(\tfrac{1}{n}-\tfrac{1}{n^2})\tfrac{n}{n-1}\varepsilon[-\lambda+\tfrac{mn-3n+2}{n-1}\varepsilon+(n-1)\delta-1]^{n-1}[-\lambda-\varepsilon+n\delta-1]^{n(n-1)}[-\lambda-\tfrac{n}{n-1}\varepsilon+(n-1)\delta-1+\tfrac{1}{n}]^{n(m-n)-1} \\
& \times[-\lambda+\tfrac{-2n+mn+1}{n-1}\varepsilon+(n-2)\delta-1]^{n}[-\lambda+\tfrac{mn-n}{n-1}\varepsilon-\delta-1]^{n(n-1)}[-\lambda-\tfrac{1}{n-1}\varepsilon-n\delta]^{n(m-n)-1}\times n(m-n).
\end{aligned}
$$

This polynomial is the sum of three parts. The first is composed of all the diagonal factors of the characteristic polynominal $\det A$, that is $[-\lambda+\frac{mn-3n+2}{n-1}\varepsilon+(n-1)\delta-1]^{n}[-\lambda-\varepsilon+n\delta-1]^{n(n-1)}[-\lambda-\frac{n}{n-1}\varepsilon+(n-1)\delta-1+\frac{1}{n}]^{n(m-n)}[-\lambda+\frac{-2n+mn+1}{n-1}\varepsilon+(n-2)\delta-1]^{n}[-\lambda+\frac{mn-n}{n-1}\varepsilon-\delta-1]^{n(n-1)}[-\lambda-\frac{1}{n-1}\varepsilon-n\delta]^{n(m-n)}$. In Table2.A, $\frac{\partial\dot{p}_{ij}}{\partial p_{ij}}$ or $\frac{\partial\dot{q_{ji}}}{\partial q_{ji}}$ for $s=i,t=j$ of each column is corresponded with each diagonal factor.

The second with the negative sign is composed of all the diagonal factors except an entry $\frac{\partial\dot{p}_{ij}}{\partial p_{st}}$ for $(i,j)\in I_1,(s,t)\in I_3,s=i,t=j$, and an entry $\frac{\partial\dot{p}_{ij}}{\partial p_{st}}$ for $(i,j)\in I_1,(s,t)\in I_3,s=i$. The value of an entry $\frac{\partial\dot{p}_{ij}}{\partial p_{st}}$ for $(i,j)\in I_1,(s,t)\in I_3,s=i,t=j$ is $-\frac{n}{n-1}\varepsilon$. The value of an entry $\frac{\partial\dot{p}_{ij}}{\partial p_{st}}$ for $(i,j)\in I_1,(s,t)\in I_3,s=i$ is $-\frac{1}{n}-\frac{2n-mn-1}{n(n-1)}\varepsilon$. The number of such terms is $n(m-n)$.

The third with the negative sign is composed of all diagonal factors except an entry $\frac{\partial\dot{p}_{ij}}{\partial p_{st}}$ for $(i,j)\in I_3,(s,t)\in I_3,t=j,s=i$ and an entry $\frac{\partial\dot{p}_{ij}}{\partial q_{ts}}$ for $(i,j)\in I_3,(s,t)\in I_3,s=i,t=j$. The value of an entry $\frac{\partial\dot{p}_{ij}}{\partial p_{st}}$ for $(i,j)\in I_3,(s,t)\in I_3,t=j,s=i$ is $\frac{1}{n}-\frac{1}{n^2}$. The value of an entry $\frac{\partial\dot{p}_{ij}}{\partial q_{ts}}$ for $(i,j)\in I_3,(s,t)\in I_3,s=i,t=j$ is $\frac{n}{n-1}\varepsilon$. The number of such terms is $n(m-n)$.

Factoring these parts and arranging, we get the characteristic polynomial as follows.

$$
\begin{aligned}
& [\lambda - \tfrac{mn-3n+2}{n-1}\varepsilon - (n-1)\delta + 1]^{n-1}[\lambda + \varepsilon - n\delta + 1]^{n(n-1)}[\lambda + \tfrac{n}{n-1}\varepsilon - (n-1)\delta + 1 - \tfrac{1}{n}]^{n(m-n)-1} \\
& \times[\lambda - \tfrac{-2n+mn+1}{n-1}\varepsilon - (n-2)\delta + 1]^{n}[\lambda - \tfrac{mn-n}{n-1}\varepsilon + \delta + 1]^{n(n-1)}[\lambda + \tfrac{1}{n-1}\varepsilon + n\delta]^{n(m-n)-1} \\
& \times\big([\lambda - \tfrac{mn-3n+2}{n-1}\varepsilon - (n-1)\delta + 1][\lambda + \tfrac{n}{n-1}\varepsilon - (n-1)\delta + 1 - \tfrac{1}{n}][\lambda + \tfrac{1}{n-1}\varepsilon + n\delta] \\
& +(-\tfrac{n}{n-1}\varepsilon)(-\tfrac{1}{n} - \tfrac{2n-mn-1}{n(n-1)}\varepsilon)[\lambda + \tfrac{n}{n-1}\varepsilon - (n-1)\delta + 1 - \tfrac{1}{n}] \times n(m-n) \\
& +(\tfrac{1}{n} - \tfrac{1}{n^2})\tfrac{n}{n-1}\varepsilon[\lambda - \tfrac{mn-3n+2}{n-1}\varepsilon - (n-1)\delta + 1] \times n(m-n)\big) \\
= \; & [\lambda - \tfrac{mn-3n+2}{n-1}\varepsilon - (n-1)\delta + 1]^{n-1} \\
& \times[\lambda + \varepsilon - n\delta + 1]^{n(n-1)} \\
& \times[\lambda + \tfrac{n}{n-1}\varepsilon - (n-1)\delta + 1 - \tfrac{1}{n}]^{n(m-n)-1} \\
& \times[\lambda - \tfrac{-2n+mn+1}{n-1}\varepsilon - (n-2)\delta + 1]^{n} \\
& \times[\lambda - \tfrac{mn-n}{n-1}\varepsilon + \delta + 1]^{n(n-1)} \\
& \times[\lambda + \tfrac{1}{n-1}\varepsilon + n\delta]^{n(m-n)-1} \\
& \times[\lambda - \tfrac{mn-n^2-1}{n-1}\varepsilon + n\delta] \\
& \times[\lambda + \tfrac{1}{n-1}(n^3 - n^2 + n - mn^2 + mn)\varepsilon + (1-n)\delta + 1 - \tfrac{1}{n}] \\
& \times[\lambda + (mn - n^2 - n + 2)\varepsilon + (1-n)\delta + 1].
\end{aligned}
$$

□

3.2 Particular-hybrid systems

When we restrict our attention to sender-receiver games with $|m - n| = 1$, we find another type of neutrally stable strategies with the same feature that characterizes an extended-signaling system. That is, a rest point near the strategy of this type can be asymptotically stable under the selection-mutation dynamics. We name such a neutrally stable strategy of this type *particular-hybrid systems* for a normal expression of the theory of signaling games (see Example 4 below). We begin by defining a particular-hybrid system.

Definition 4 Suppose $|m - n| = 1$. We say that a pair of strategies $(P^\star, Q^\star) \in \mathcal{P}^{\triangle}_{n\times m} \times \mathcal{Q}^{\triangle}_{m\times n}$ is a *particular*-hybrid signaling system if the following properties hold:

whenever $n \leq m$, there exist sets $L \subset M$ and $K \subset N$ such that

(i) $|L| = |K| = 2$,

(ii) $p^\star_{ij} = q^\star_{ji} = 1$ or $p^\star_{ij} = q^\star_{ji} = 0$ for each $i \in N \backslash K$ and $j \in M \backslash L$,

(iii) $p^\star_{ik} = \frac{1}{2}$ and $q^\star_{ki}$=1 for each $k \in K$;

whenever $n > m$, there exist sets $L \subset M$ and $K \subset N$ such that

(i) $|L| = |K| = 2$,

(ii) $q^\star_{ij} = q^\star_{ji} = 1$ or $p^\star_{ij} = q^\star_{ji} = 0$ for each $k \in M \backslash L$ and $j \in N \backslash K$,

(iii) $q^\star_{jl} = \frac{1}{2}$ and $p^\star_{lj}$=1 for each $l \in L$;

where $\left(p^\star_{ij}, q^\star_{ji}\right)_{(i,j)\in N\times M}$ denote the entries of the particular-hybrid system $(P^\star, Q^\star)$.

Example 4. A particular-hybrid sytem:

$$P^\star_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{pmatrix}, \; Q^\star_4 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

We find a rest point of the selection-mutation dynamics near each particular-hybrid system. In general, a rest point of our dynamical system, $S' = \Phi(S)$, is defined as a point that satisfies $\Phi\left((p_{ij}, q_{ji})_{(i,j)\in N\times M}\right) = \mathbf{0}$, where $\mathbf{0}$ is a zero-column vector. Especially, we focus on a rest point called to be symmetric for the corresponding particular-hybrid system.

Definition 5. Let $(P^\star, Q^\star)$ be a particular-hybrid system. We say that a rest point $(\tilde{P}(\varepsilon, \delta), \tilde{Q}(\varepsilon, \delta)) \in \mathbf{S}$ has a symmetric form for $(P^\star, Q^\star)$ if

· For $n \leq m$, for some real values $\varepsilon_1, \varepsilon_2, \varepsilon, \delta_1, \delta_2, \delta_3$,

$$\tilde{p}_{ij}(\varepsilon, \delta) = \begin{cases} 1 - (m-3)\varepsilon_1 - 2\varepsilon_2 & \text{for each } p^\star_{ij} = 1, \\ \varepsilon_1 & \text{for each } p^\star_{ij} = 0 \text{ with } j \neq k, \\ \varepsilon_2 & \text{for each } p^\star_{ij} = 0 \text{ with } j = k, \\ \frac{1}{m-2}\varepsilon_3 & \text{for each } p^\star_{ij} = 0 \text{ with } p^\star_{ik} = \frac{1}{2} \\ \frac{1}{2} - \frac{1}{2}\varepsilon_3 & \text{for each } p^\star_{ij} = \frac{1}{2}. \end{cases}$$

$$\tilde{q}_{ji}(\varepsilon, \delta) = \begin{cases} 1 - (n-2)\delta_1 - \delta_2 & \text{for each } q^\star_{ji} = 1 \text{ with } j \neq k, \\ \delta_1 & \text{for each } q^\star_{ji} = 0 \text{ with } j \neq k \text{ and } q^\star_{ki} \neq 1, \\ \delta_2 & \text{for each } q^\star_{ji} = 0 \text{ with } j \neq k \text{ and } q^\star_{ki} = 1, \\ \delta_3 & \text{for each } q^\star_{ki} = 0, \\ 1 - (n-1)\delta_3 & \text{for each } q^\star_{ki} = 1. \end{cases}$$

where $\big(\tilde{p}_{ij}(\varepsilon, \delta), \tilde{q}_{ji}(\varepsilon, \delta)\big)_{(i,j) \in N \times M}$ are entries of $(\tilde{P}(\varepsilon, \delta), \tilde{Q}(\varepsilon, \delta))$.

For the particular-hybrid sytem in Example 4, the symmetric rest point of the selection-mutation dynamics has a form presented in the following Example 5.

Example 5.

$$\tilde{P}_4(\varepsilon, \delta) = \begin{pmatrix} \varepsilon_4 & 1 - \varepsilon_1 - 2\varepsilon_2 & \varepsilon_1 & \varepsilon_2 \\ \frac{1}{2} - \frac{1}{2}\varepsilon_3 & \frac{1}{2}\varepsilon_3 & \frac{1}{2}\varepsilon_3 & \frac{1}{2} - \frac{1}{2}\varepsilon_3 \\ \varepsilon_2 & \varepsilon_1 & 1 - \varepsilon_1 - 2\varepsilon_2 & \varepsilon_2 \end{pmatrix},$$

$$\tilde{Q}_4(\varepsilon,\delta) = \begin{pmatrix} \delta_3 & 1-2\delta_3 & \delta_3 \\ 1-\delta_1-\delta_2 & \delta_2 & \delta_1 \\ \delta_1 & \delta_2 & 1-\delta_1-\delta_2 \\ \delta_3 & 1-2\delta_3 & \delta_3 \end{pmatrix}.$$

Theorem 3. Let $(P^\star, Q^\star) \in \mathcal{P}^\triangle_{n\times m} \times \mathcal{Q}^\triangle_{m\times n}$ be a paticular-hybrid system. Then, for each pair of mutation rates, (ε,δ), there exists a neighborhood of the point $(P^\star, Q^\star)$ that contains a unique symmetric rest point, $(\tilde{P}(\varepsilon,\delta), \tilde{Q}(\varepsilon,\delta))$, of the selection-mutation dynamics.

Proof. We consider the case $n \leq m$ and abbreviate the proof for the case $n > m$ because it is similar to that of the former.

We now find the values of the entries of the symmetric rest point, $\varepsilon_1, \varepsilon_2, \varepsilon_3$, and $\delta_1, \delta_2, \delta_3$ consistent with the conditions required for the rest point, $\dot{p} = 0$ and $\dot{q} = 0$.

We write down our dynamical system $S' = \Phi(S)$ of the selection-mutation dynamics:

$$\begin{cases} \dot{p}_{11} &= p_{11}(q_{11} - p_{11}q_{11} - p_{12}q_{21} - \cdots - p_{1m}q_{m1}) + \varepsilon(1 - mp_{11}), \\ \dot{p}_{12} &= p_{12}(q_{21} - p_{11}q_{11} - p_{12}q_{21} - \cdots - p_{1m}q_{m1}) + \varepsilon(1 - mp_{12}), \\ &\vdots \\ \dot{p}_{nm} &= p_{nm}(q_{mn} - p_{n1}q_{1n} - p_{n2}q_{2n} - \cdots - p_{nm}q_{mn}) + \varepsilon(1 - mp_{nm}), \\ \dot{q}_{11} &= q_{11}(p_{11} - q_{11}p_{11} - q_{12}p_{21} - \cdots - q_{1n}p_{n1}) + \delta(1 - nq_{11}), \\ \dot{q}_{12} &= q_{12}(p_{21} - q_{11}p_{11} - q_{12}p_{21} - \cdots - q_{1n}p_{n1}) + \delta(1 - nq_{12}), \\ &\vdots \\ \dot{q}_{mn} &= q_{mn}(p_{nm} - q_{m1}p_{1m} - q_{m2}p_{2m} - \cdots - q_{mn}p_{nm}) + \delta(1 - nq_{mn}). \end{cases}$$

By removing equations, $\dot{p}_{ij} = 0$ with $p^\star_{ij} = 1$, $p^\star_{ij} = \varepsilon_3$ and $\dot{q}_{ij} = 0$ with

$q^\star_{ji} = 1$, from our whole system, $\Phi(S) = \mathbf{0}$, we obtain a following sub-system of $\Phi(S) = \mathbf{0}$, which is denoted by $F = \mathbf{0}$.

$$F = \mathbf{0} \Leftrightarrow \begin{cases} \dot{p}_{ij} = 0 \ \text{ for each } p^\star_{ij} = 0 \text{ with } j \neq k, \\ \dot{p}_{ij} = 0 \ \text{ for each } p^\star_{ij} = 0 \text{ with } j = k, \\ \dot{p}_{ij} = 0 \ \text{ for each } p^\star_{ij} = \frac{1}{2}, \\ \dot{q}_{ji} = 0 \ \text{ for each } q^\star_{ji} = 0 \text{ with } j \neq k \text{ and } q^\star_{ki} \neq 1, \\ \dot{q}_{ji} = 0 \ \text{ for each } q^\star_{ji} = 0 \text{ with } j \neq k \text{ and } q^\star_{ki} = 1, \\ \dot{q}_{ji} = 0 \ \text{ for each } q^\star_{ki} = 0. \end{cases}.$$

By substituting the entries $(\tilde{p}_{ij}, \tilde{q}_{ji})$ of Definition 5 into the sub-system $F = \mathbf{0}$, we get a following reduced system, $f(\varepsilon_1, \varepsilon_2, \varepsilon_3, \delta_1, \delta_2, \delta_3; \varepsilon, \delta) = \mathbf{0}$, consisting of six equations, $f_i(\varepsilon_1, \varepsilon_2, \varepsilon_3, \delta_1, \delta_2, \delta_3; \varepsilon, \delta) = 0, i = 1, 2, 3, 4, 5, 6.$

$$F(\varepsilon_1, \varepsilon_2, \varepsilon_3, \delta_1, \delta_2, \delta_3; \varepsilon, \delta) = \mathbf{0} \Leftrightarrow \begin{cases} \tilde{p}_{ij}[\tilde{q}_{ij} - \sum_{(i,s)\in N\times M} \tilde{p}_{is}\tilde{q}_{sj}] + \varepsilon(1 - m\tilde{p}_{ij}) = 0 \ \text{ for each } p^\star_{ij} = 0 \text{ with } j \neq k, \\ \tilde{p}_{ij}[\tilde{q}_{ij} - \sum_{(i,s)\in N\times M} \tilde{p}_{is}\tilde{q}_{sj}] + \varepsilon(1 - m\tilde{p}_{ij}) = 0 \ \text{ for each } p^\star_{ij} = 0 \text{ with } j = k, \\ \tilde{p}_{ij}[\tilde{q}_{ij} - \sum_{(i,s)\in N\times M} \tilde{p}_{is}\tilde{q}_{sj}] + \varepsilon(1 - m\tilde{p}_{ij}) = 0 \ \text{ for each } p^\star_{ij} = \dfrac{1}{2}, \\ \tilde{q}_{ji}[\tilde{p}_{ij} - \sum_{(t,j)\in N\times M} \tilde{q}_{jt}\tilde{p}_{tj}] + \delta(1 - n\tilde{q}_{ji}) = 0 \ \text{ for each } q^\star_{ji} = 0 \text{ with } j \neq k \text{ and } q^\star_{ki} \neq 1, \\ \tilde{q}_{ji}[\tilde{p}_{ij} - \sum_{(t,j)\in N\times M} \tilde{q}_{jt}\tilde{p}_{tj}] + \delta(1 - n\tilde{q}_{ji}) = 0 \ \text{ for each } q^\star_{ji} = 0 \text{ with } j \neq k \text{ and } q^\star_{ki} = 1, \\ \tilde{q}_{ji}[\tilde{p}_{ij} - \sum_{(t,j)\in N\times M} \tilde{q}_{jt}\tilde{p}_{tj}] + \delta(1 - n\tilde{q}_{ji}) = 0 \ \text{ for each } q^\star_{ki} = 0. \end{cases}.$$

$$\Leftrightarrow \begin{cases} \varepsilon_1 \{\delta_1 - (m-3)\varepsilon_1\delta_1 - [1 - (m-3)\varepsilon_1 - 2\varepsilon_2][1 - (n-2)\delta_1 - \delta_2] - 2\varepsilon_2\delta_3\} + \varepsilon(1 - m\varepsilon_1) = 0, \\ \varepsilon_2 \{\delta_2 - (m-3)\varepsilon_1\delta_1 - [1 - (m-3)\varepsilon_1 - 2\varepsilon_2][1 - (n-2)\delta_1 - \delta_2] - 2\varepsilon_2\delta_3\} + \varepsilon(1 - m\varepsilon_2) = 0, \\ (\frac{1}{2} - \frac{1}{2}\varepsilon_3)\left\{1 - (n-1)\delta_3 - 2(\frac{1}{2} - \frac{1}{2}\varepsilon_3)[1 - (n-1)\delta_3] - \varepsilon_3\delta_2\right\} + \varepsilon\left[1 - m(\frac{1}{2} - \frac{1}{2}\varepsilon_3)\right] = 0, \\ \delta_1\left\{\varepsilon_1 - (n-2)\delta_1\varepsilon_1 - \frac{1}{m-2}\varepsilon_3\delta_2 - [1 - \delta_2 - (n-2)\delta_1][1 - (m-3)\varepsilon_1 - 2\varepsilon_2]\right\} + \delta(1 - n\delta_1) = 0. \\ \delta_2\left\{\frac{1}{m-2}\varepsilon_3 - [1 - \delta_2 - (n-2)\delta_1][1 - (m-3)\varepsilon_1 - 2\varepsilon_2] - (n-2)\varepsilon_1\delta_1 - \frac{1}{m-2}\varepsilon_3\delta_2\right\} + \delta(1 - n\delta_1) = 0. \\ \delta_3\left\{\varepsilon_2 - [1 - (n-1)\delta_3]\left[\frac{1}{2} - \frac{1}{2}\varepsilon_3\right] - (n-1)\varepsilon_2\delta_3\right\} + \delta(1 - n\delta_3) = 0. \end{cases}$$

From this, we see that $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \delta_1, \delta_2, \delta_3; \varepsilon, \delta) = (0,0,0,0,0,0;0,0)$ is a solution to the reduced system, i.e., $f_i(0,0,0,0,0,0;0,0) = 0, i = 1,2,3,4,5,6$. Let Df denote the Jacobian matrix of $f_1, f_2, f_3, f_4, f_5, f_6$ with respect to $\varepsilon_1, \varepsilon_2, \varepsilon_3, \delta_1, \delta_2, \delta_3$, that is,

$$Df = \begin{pmatrix} \frac{\partial f_1}{\partial \varepsilon_1} & \frac{\partial f_1}{\partial \varepsilon_2} & \frac{\partial f_1}{\partial \varepsilon_3} & \frac{\partial f_1}{\partial \delta_1} & \frac{\partial f_1}{\partial \delta_2} & \frac{\partial f_1}{\partial \delta_3} \\ \frac{\partial f_2}{\partial \varepsilon_1} & \frac{\partial f_2}{\partial \varepsilon_2} & \frac{\partial f_2}{\partial \varepsilon_3} & \frac{\partial f_2}{\partial \delta_1} & \frac{\partial f_2}{\partial \delta_2} & \frac{\partial f_2}{\partial \delta_3} \\ \frac{\partial f_3}{\partial \varepsilon_1} & \frac{\partial f_3}{\partial \varepsilon_2} & \frac{\partial f_3}{\partial \varepsilon_3} & \frac{\partial f_3}{\partial \delta_1} & \frac{\partial f_3}{\partial \delta_2} & \frac{\partial f_3}{\partial \delta_3} \\ \frac{\partial f_4}{\partial \varepsilon_1} & \frac{\partial f_4}{\partial \varepsilon_2} & \frac{\partial f_4}{\partial \varepsilon_3} & \frac{\partial f_4}{\partial \delta_1} & \frac{\partial f_4}{\partial \delta_2} & \frac{\partial f_4}{\partial \delta_3} \\ \frac{\partial f_5}{\partial \varepsilon_1} & \frac{\partial f_5}{\partial \varepsilon_2} & \frac{\partial f_5}{\partial \varepsilon_3} & \frac{\partial f_5}{\partial \delta_1} & \frac{\partial f_5}{\partial \delta_2} & \frac{\partial f_5}{\partial \delta_3} \\ \frac{\partial f_6}{\partial \varepsilon_1} & \frac{\partial f_6}{\partial \varepsilon_2} & \frac{\partial f_6}{\partial \varepsilon_3} & \frac{\partial f_6}{\partial \delta_1} & \frac{\partial f_6}{\partial \delta_2} & \frac{\partial f_6}{\partial \delta_3} \end{pmatrix}.$$

At the point $(\varepsilon_1, \varepsilon_2, \delta_1; \varepsilon, \delta) = (0,0,0;0,0)$, we have

$$\det(Df(\mathbf{0})) = \begin{vmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} \neq 0,$$

□

Corollary 2. The first-order approximated entries $\big(\tilde{p}_{ij}(\varepsilon, \delta), \tilde{q}_{ji}(\varepsilon, \delta)\big)_{(i,j) \in N \times M}$ of a rest point $(\tilde{P}(\varepsilon, \delta), \tilde{Q}(\varepsilon, \delta)) \in S$ in the neighborhood of a particular-hybrid system $(P^\star, Q^\star)$ is explicitly given as follows:

· For $n \leq m$,

$$\tilde{p}_{ij}(\varepsilon,\delta) = \begin{cases} 1-(m-1)\varepsilon & \text{for each } p^\star_{ij}=1, \\ \varepsilon & \text{for each } p^\star_{ij}=0 \text{ with } j\neq k, \\ \varepsilon & \text{for each } p^\star_{ij}=0 \text{ with } j=k, \\ \varepsilon & \text{for each } p^\star_{ij}=0 \text{ with } p^\star_{ik}=\frac{1}{2} \\ \frac{1}{2}-\frac{m-2}{2}\varepsilon & \text{for each } p^\star_{ij}=\frac{1}{2} \text{ with } j=k. \end{cases}$$

$$\tilde{q}_{ji}(\varepsilon,\delta) = \begin{cases} 1-(n-1)\delta & \text{for each } q^\star_{ji}=1 \text{ with } j\neq k, \\ \delta & \text{for each } q^\star_{ji}=0 \text{ with } j\neq k \text{ and } q^\star_{ki}\neq 1, \\ \delta & \text{for each } q^\star_{ji}=0 \text{ with } j\neq k \text{ and } q^\star_{ki}=1, \\ 2\delta & \text{for each } q^\star_{ji}=0 \text{ with } j=k, \\ 1-2(n-1)\delta & \text{for each } q^\star_{ki}=1 \text{ with } j=k. \end{cases}$$

Proof. Since the proof is similar to that of Corollary 1, we omit it. □

Example 6.

The first order approximation of the symmetric rest point close to the particular-hybrid system $(P^\star_4, Q^\star_4), i=1,2$, in Example 5, is given by

$$\tilde{P}_4(\varepsilon,\delta) = \begin{pmatrix} \varepsilon & 1-3\varepsilon & \varepsilon & \varepsilon \\ \frac{1}{2}-\varepsilon & \varepsilon & \varepsilon & \frac{1}{2}-\varepsilon \\ \varepsilon & \varepsilon & 1-3\varepsilon & \varepsilon \end{pmatrix}, \tilde{Q}_4(\varepsilon,\delta) = \begin{pmatrix} 2\delta & 1-4\delta & 2\delta \\ 1-2\delta & \delta & \delta \\ \delta & \delta & 1-2\delta \\ 2\delta & 1-4\delta & 2\delta \end{pmatrix}.$$

By deriving the characteristic equation of the Jacobian matrix evaluated at the first-order approximated rest point close to a particular-hybrid system, we get the following result.[14]

Theorem 4. If $n \leq m$ and $\frac{\varepsilon}{\delta} > \frac{n-1}{2}$ for sufficiently small ε and δ, then the symmetric rest point close to a paticular-hybrid system is asymptotically stable.

Proof. Following the similar procedure in the proof of Theorem 2, the characteristic equation of the first order approximated Jacobian matrix evaluated at the symmetric rest point close to any particular-hybrid system (with $n \leq m$) is given by

$$\begin{aligned}
&[\lambda-(m-2)\varepsilon-2(n-1)\delta+1]^{(m-2)}\\
&\times[\lambda+\varepsilon-n\delta+1]^{(m-3)(n-1)}\\
&\times[\lambda+\varepsilon-(n+1)\delta+1]^{(2(n-1))}\\
&\times[\lambda+2\varepsilon-(2n-1)\delta+1]^{(m-2)}\\
&\times[\lambda-\tfrac{m-6}{2}\varepsilon-(n-1)\delta+\tfrac{1}{2}]^{2}\\
&\times[\lambda-(m-1)\varepsilon-(n-2)\delta+1]^{n-1}\\
&\times[\lambda-m\varepsilon+\delta+1]^{(m-2)(n-2)}\\
&\times[\lambda-m\varepsilon+\delta+1]^{m-2}\\
&\times[\lambda-\tfrac{m}{2}\varepsilon+\delta+\tfrac{1}{2}]^{2(n-1)}\\
&\times[\lambda-\tfrac{m-2}{2}\varepsilon-(n-2)\delta+\tfrac{1}{2}]^{2}\\
&\times[\lambda+\tfrac{2-m}{2}\varepsilon+(2-3n)\delta+\tfrac{1}{2}]\\
&\times[\lambda+(4-m)\varepsilon+(n+1)\delta+1]\\
&\times[\lambda+2\varepsilon+(1-n)\delta]=0
\end{aligned}$$

where λ is the eigenvalue. □

[14]We abbreviate the result for $n > m$, since it is similar.

3.3 Other systems

We show that a rest point close to most of other systems except for an extended-signaling system and a particular-hybrid system is not asymptotically stable.

Theorem 5.

A rest point close to all strategies that have at least one of following properties is not asymptotically stable.

(i) There exist some $K \subset M$, $|K| = k \geq 3$ and some $i \in N$ such that for each $j \in K$, $0 < p_{ij} < 1$ with $\sum_{j \in K} p_{ij} = 1$ and $q_{ji} = 1$.

(ii) There exist some $L \subset N$, $|L| = l \geq 3$ and some $j \in M$ such that for each $i \in L$, $0 < q_{ji} < 1$ with $\sum_{j \in L} q_{ji} = 1$ and $p_{ij} = 1$.

(iii) There exist some $K \subset M$, $|K| = k \geq 2$ and some $i \in N$ such that for each $j \in K$, $0 < p_{ij} < 1$ with $\sum_{j \in K} p_{ij} = 1$ and $q_{ji} = 1$,

and there exist some $L \subset N$, $|L| = l \geq 2$ and some $j \in M$ such that for each $i \in L$, $0 < q_{ji} < 1$ with $\sum_{j \in L} q_{ji} = 1$ and $p_{ij} = 1$.

(iv) There exists some $w \in W$, $|W| \geq 2$ such that there exist some $K_w \subset M$, $|K_w| \geq 2$ and some $i \in N$ with for each $j \in K_w$, $0 < p_{ij} < 1$, $\sum_{j \in K_w} p_{ij} = 1$ and $q_{ji} = 1$,

or there exists some $v \in V$, $|V| \geq 2$ such that there exist some $L_v \subset N$, $|L_v| \geq 2$ and some $j \in M$ with for each $i \in L_v$, $0 < q_{ji} < 1$, $\sum_{j \in L_v} q_{ij} = 1$ and $p_{ij} = 1$.

(v) There exist some $L \subset N$, $|L| = l \geq 2$ and some $K \subset M$, $|K| = k = l$ such that for each $j \in K$, $0 < p_{ij} < 1$ with $\sum_{j \in K} p_{ij} = 1$ and for each $i \in L$, $0 < q_{ji} < 1$ with $\sum_{i \in L} q_{ji} = 1$.

Proof.

Let $J\Phi$ denote the Jacobian matrix at a rest point of the selection-mutation dynamics. The characteristic equation is given by $\det(J\Phi - \lambda I) = 0$, where λ is the eigenvalue. Let λ^t denote the tth-order term of this equation and d_t denote the coefficient of the term. If the rest point is asymptotically stable, the sign of any solution, λ, of this equation must be negative. The LHS of this equation can be factorizing as the following form

$$\prod_{s=1}^{2MN} (\lambda + a_s + b_s\varepsilon + c_s\delta) = 0.$$

where a_s, b_s and c_s are some real numbers. If any solution, λ, of this equation is negative, then each coefficient d_t that does not include ε or δ must be positive [15]. To the contrary, we will show that there exists some negative coefficient d_t that does not include ε or δ[16].

First, we make clear the relation between the rest point under the replicator dynamics and the rest point under the selection-mutation dynamics. If a rest point $(\bar{P}, \bar{Q})$ under the replicator dynamics is not Nash equilibrium, then there exists no rest point of the selection-mutation dynamcs near $(\bar{P}, \bar{Q})$ for sufficiently small ε, δ (Hofbauer and Hatteger, 2008). Therefore, we consider all Nash strategies under the replicator dynamics. The Nash strategy in sender-receiver games has the following properties. Let $H \subset N \cap M$ denote the common subset of N and M. Let $L \subset N$ and $K \subset M$ denote each subset of N and M.

[15]It is true in the case $t = 0$. It is not asymptotically stable if the constant of this equation is negative.

[16]In short, we will show that there exists some negative d_t when ε and δ go to zero.

(1) $p_{ij} = q_{ji} = 1$ or $p_{ij} = q_{ji} = 0$ for some $i \in N$ and each $j \in M$,
or $q_{ji} = p_{ij} = 1$ or $q_{ji} = p_{ij} = 0$ for some $j \in M$ and each $i \in N$.
(2) $0 < p_{ij} < 1$ with $\sum_{j \in K} p_{ij} = 1$ and $q_{ji} = 1$, for each $j \in K, |K| \geq 2$ and some $i \in N$,
or $p_{ij} = 1$ and $0 < q_{ji} < 1$ with $\sum_{i \in L} q_{ji} = 1$, for each $i \in L, |L| \geq 2$ and some $j \in M$.
(3) There exist some $K \subset M$ and some $L \subset N$, $|K| = |L|$ such that $0 < p_{ij} < 1$ and $0 < q_{ji} < 1$, for each $i \in L$ and some $j \in K$,
(4) $0 < p_{ij} < 1$ with $\sum_{j \in K} p_{ij} = 1$ and $q_{ji} = 0$, for each $i \in N$ and some $i \in N$,
or $p_{ij} = 0$ and $0 < q_{ji} < 1$ with $\sum_{i \in N} q_{ji} = 1$, for each $i \in N$ and some $j \in M$.

We examine whether the characteristic equation has some negative coefficient d^t when the Nash strategy has one or some of the above properties.

The rest point of the strategy that only satisfies (1) is asymptotically stable because it is an evolutionarily stable strategy.

Conversely, under the replicator dynamics, the rest point of the strategy that has (4) is not asymptotically stable because there exists 0 diagonal factors of Jacobian matrix of this rest point [17].

However, under the selection-mutation dynamics, there can be non-zero factors of Jacobian matrix of the rest point close to the strategy that includes (4) [18].

We basically consider the strategy that only has property (1) , and then

[17] In this case (4), $\frac{\partial \dot{q}_{ji}}{\partial q_{ji}} = p_{ij} - p_{ij}q_{ji} - \sum_{t \in N} p_{ti}q_{jt} = 0$

[18] For example, the extended signaling system doesn't have 0 diagonal factor though it is the strategy that satisfies (4).

consider each strategy that has the property (1) and (2), (1) and (3), and (1), (2) and (3). Seuquentially, We focus on the column of $\mathrm{J}\Phi - \lambda$ and analyze each column's effect on the stability of the rest point of the strategy that has (2) and (3). We disregard (4) because (4) doesn't effect on the sign of coefficient d_t.

Case (i) There exist some $K \subset M$, $|K| = k \geq 3$ and some $i \in N$ such that for each $j \in K$, $0 < p_{ij} < 1$ with $\sum_{j \in K} p_{ij} = 1$ and $q_{ji} = 1$.

We consider that the strategy is composed of (1) and (2). This strategy satisfies $|M| - |N| = |K| - 1$. It depends on the value of p_{ij} for $i \in N$ and $j \in K$ wheather this system has the rest point. This system has the rest point in the case that $p_{ij} = \frac{1}{k}$ for $i \in N$ and $j \in K$[19] .

Case (i).

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{1}{k} & \cdots & \frac{1}{k} \end{pmatrix}, Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \vdots \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In this case , we have the following factors a_{IJ} and a_{JI} of the Jacobian matrix, focusing $\dot{p}_{ij}, \dot{q}_{ji}$ for some $i, \in N$ and for each $j \in K$.

[19]We abbreviate this proof because this proof is similar to Theorem 3.

$\dot{p}_{ij}, \dot{q}_{ji}$	the range of i,j,s,t	factors a_{IJ} of the Jacobian matrix	factors a_{JI} of the Jacobian matrix
(1)	$s=i, t=j$	$\frac{\partial \dot{p}_{ij}}{\partial p_{st}} = q_{ji} - p_{ij}q_{ji} - \sum_{j\in k} p_{ij}q_{ji}$	
(2)	$s=i, t\in K, t\neq j$	$\frac{\partial \dot{p}_{ij}}{\partial p_{st}} = -p_{ij}q_{ti}$	$\frac{\partial \dot{p}_{it}}{\partial p_{ij}} = -p_{it}q_{ji}$
(3)	$s=i, t\notin K$	$\frac{\partial \dot{p}_{ij}}{\partial p_{st}} = 0$	$\frac{\partial \dot{p}_{il}}{\partial p_{ij}} = -p_{it}q_{ji}$
(4)	$s\neq i, t\neq j, t\in M$	$\frac{\partial \dot{p}_{ij}}{\partial p_{st}} = 0$	$\frac{\partial \dot{p}_{st}}{\partial p_{ij}} = 0$
(5)	$s=i, t=j$	$\frac{\partial \dot{p}_{ij}}{\partial q_{ts}} = p_{ij} - p_{ij}^2$	$\frac{\partial \dot{q}_{ji}}{\partial p_{ij}} = q_{ji} - q_{ji}^2 = 0$
(6)	$s=i, t\neq j, t\in K$	$\frac{\partial \dot{p}_{ij}}{\partial q_{ti}} = -p_{ij}p_{it}$	$\frac{\dot{q}_{ti}}{\partial p_{ij}} = 0$
(7)	$s=i, t\neq j, t\notin K$	$\frac{\partial \dot{p}_{ij}}{\partial q_{ts}} = 0$	$\frac{\partial \dot{q}_{ti}}{\partial p_{ij}} = 0$
(8)	$s\neq i$	$\frac{\partial \dot{p}_{ij}}{\partial q_{ts}} = 0$	$\frac{\partial \dot{q}_{js}}{\partial p_{ij}} = 0$

$\dot{q}_{ts}, \dot{p}_{st}$	the range of $i,j,s,t\in M$	factors a_{IJ} of the Jacobian matrix	factors a_{JI} of the Jacobian matrix
(9)	$s=i, t=j$	$\frac{\partial \dot{q}_{ji}}{\partial q_{ts}} = p_{ij} - q_{ji}p_{ij} - \sum_{i\in N} q_{ji}p_{ij}$	
(10)	$s\in L, s\neq i$	$\frac{\partial \dot{q}_{ji}}{\partial q_{ts}} = -q_{ji}p_{sj}$	$\frac{\partial \dot{q}_{js}}{\partial q_{ji}} = -q_{js}p_{ij} = 0$
(11)	$s\notin L$	$\frac{\partial \dot{q}_{ji}}{\partial q_{js}} = 0$	$\frac{\partial \dot{q}_{js}}{\partial q_{ji}} = 0$
(12)	$s=i, t=j$	$\frac{\partial \dot{q}_{ji}}{\partial p_{st}} = q_{ji} - q_{ji}^2 = 0$	$\frac{\partial \dot{p}_{ij}}{\partial q_{ji}} = p_{ij} - p_{ij}^2$
(13)	$s=i, t\in K, t\neq j$	$\frac{\partial \dot{q}_{ji}}{\partial p_{st}} = 0$	$\frac{\partial \dot{p}_{st}}{\partial q_{ji}} = 0$
(14)	$s=i, t\neq j, t\notin K$	$\frac{\partial \dot{q}_{ji}}{\partial p_{st}} = 0$	$\frac{\partial \dot{p}_{st}}{\partial q_{ji}} = 0$
(15)	$s\neq i, t=j$	$\frac{\partial \dot{q}_{ji}}{\partial p_{st}} = 0$	$\frac{\partial \dot{p}_{st}}{\partial q_{ji}} = 0$
(16)	$s\neq i, t\neq j, t\notin K$	$\frac{\partial \dot{q}_{ji}}{\partial p_{sj}} = 0$	$\frac{\partial \dot{p}_{sj}}{\partial q_{ji}} = 0$

We consider the columns of $\frac{\partial \dot{p}_{ij}}{\partial p_{st}}, \frac{\partial \dot{p}_{ij}}{\partial q_{ts}}$, for each $s\in N$, for each $t\in M$ and $\frac{\partial \dot{q}_{ji}}{\partial p_{st}}, \frac{\partial \dot{q}_{ij}}{\partial q_{ts}}$, for each $s\in N$, for each $t\in M$. In the case k is even, the characteristic equation of the first-order approximated Jacobian matrix evaluated at the rest point close to this system is given by

$$\begin{aligned}
& (\lambda+1)^{(n-1)(2n+2+k)}(\lambda+\tfrac{1}{k})^{(n+1)k} \\
& + \sum_{r=1}^{\frac{k}{2}}(-1)^r \tfrac{k!}{2r!(k-2r)!}(\lambda+1)^{(n-1)(2n+2+k)}(\lambda+\tfrac{1}{k})^{(n+1)k}(\lambda+\tfrac{1}{k})^{-2r}(\tfrac{1}{k})^{2r} \\
= \; & (\lambda+1)^{(n-1)(2n+2+k)}(\lambda+\tfrac{1}{k})^{(n+1)k}\Big(1+\sum_{r=1}^{\frac{k}{2}}(-1)^r\tfrac{k!}{2r!(k-2r)!}(\lambda+\tfrac{1}{k})^{-2r}(\tfrac{1}{k})^{2r}\Big) \\
= \; & 0.
\end{aligned}$$

In the case k is odd, it is given by

$$(\lambda+1)^{(n-1)(2n+2+k)}(\lambda+\tfrac{1}{k})^{(n+1)k}$$
$$+\sum_{r=1}^{\frac{k-1}{2}}(-1)^r\frac{k!}{2r!(k-2r)!}(\lambda+1)^{(n-1)(2n+2+k)}(\lambda+\tfrac{1}{k})^{(n+1)k}(\lambda+\tfrac{1}{k})^{-2r}(\tfrac{1}{k})^{2r}=0.$$

These equations have negative coefficient d_t of λ or negative constant in $k \geq 3$. We turn to the general case of p_{ij} for $i \in N$ and $j \in K$. We can find negative coefficients in each case.

Case (ii): There exist some $L \subset N$, $|L| = l \geq 3$ and some $j \in M$ such that for each $i \in L$, $0 < q_{ji} < 1$ with $\sum_{j \in L} q_{ji} = 1$ and $p_{ij} = 1$.

This case is similiar to Case (i). Then Case (ii) necessarily has an negative coefficient d_t.

Case (iii): There exist some $K \subset M$, $|K| = k \geq 2$ and some $i \in N$ such that for each $j \in K$, $0 < p_{ij} < 1$ with $\sum_{j \in K} p_{ij} = 1$ and $q_{ji} = 1$.
and there are some $L \subset N$, $|L| = l \geq 2$ and some $j \in M$ such that for each $i \in L$, $0 < q_{ji} < 1$ with $\sum_{j \in L} q_{ji} = 1$ and $p_{ij} = 1$.

We consider that the strategy is composed of (1), (2) and (3).

Case (iii).

$$P=\begin{pmatrix}1&0&0&0&\cdots&0&0\\0&\ddots&0&0&\cdots&0&0\\0&0&1&0&\cdots&0&0\\0&0&0&\frac{1}{k}&\cdots&\frac{1}{k}&0\\0&0&0&0&\cdots&0&1\\0&\ddots&0&0&\cdots&0&\vdots\\0&0&0&0&\cdots&0&1\end{pmatrix},\ Q=\begin{pmatrix}1&0&0&0&0&\cdots&0\\0&\ddots&0&0&0&\cdots&0\\0&0&1&0&0&\cdots&0\\0&0&0&1&0&\cdots&0\\0&0&0&\vdots&0&\cdots&0\\0&0&0&1&0&\cdots&0\\0&0&0&0&\frac{1}{l}&\cdots&\frac{1}{l}\end{pmatrix}.$$

In the case k and l are even, the characteristic equation of the first-order approximated Jacobian matrix evaluated at the rest point close to this system is given by

$$
\begin{aligned}
& (\lambda+1)^{2m^2-2mk-m+n}(\lambda+\tfrac{1}{k})^{nk}(\lambda+\tfrac{1}{l})^{lm} \\
& +\sum_{r=1}^{\frac{k}{2}}(-1)^r\tfrac{k!}{2r!(k-2r)!}(\lambda+1)^{2m^2-2mk-m+n}(\lambda+\tfrac{1}{k})^{nk}(\lambda+\tfrac{1}{l})^{lm}(\lambda+\tfrac{1}{k})^{-2r}(\tfrac{1}{k})^{2r} \\
& +\sum_{s=1}^{\frac{l}{2}}(-1)^s\tfrac{s!}{2s!(l-2s)!}(\lambda+1)^{2m^2-2mk-m+n}(\lambda+\tfrac{1}{k})^{nk}(\lambda+\tfrac{1}{l})^{lm}(\lambda+\tfrac{1}{l})^{-2s}(\tfrac{1}{l})^{2s} \\
& +\sum_{r=1}^{\frac{k}{2}}(-1)^r\tfrac{k!}{2r!(k-2r)!}\sum_{s=1}^{\frac{l}{2}}(-1)^s\tfrac{l!}{2s!(l-2s)!} \\
& \times(\lambda+1)^{2m^2-2mk-m+n}(\lambda+\tfrac{1}{k})^{nk}(\lambda+\tfrac{1}{l})^{lm}(\lambda+\tfrac{1}{k})^{-2r}(\tfrac{1}{k})^{2r}(\lambda+\tfrac{1}{l})^{-2s}(\tfrac{1}{l})^{2s} \\
= \;& (\lambda+1)^{2m^2-2mk-m+n}(\lambda+\tfrac{1}{k})^{nk}(\lambda+\tfrac{1}{l})^{lm} \\
& \Big(1+\sum_{r=1}^{\frac{k}{2}}(-1)^r\tfrac{k!}{2r!(k-2r)!}(\lambda+\tfrac{1}{k})^{-2r}(\tfrac{1}{k})^{2r} \\
& +\sum_{s=1}^{\frac{l}{2}}(-1)^s\tfrac{s!}{2s!(l-2s)!}(\lambda+\tfrac{1}{l})^{-2s}(\tfrac{1}{l})^{2s} \\
& +\sum_{r=1}^{\frac{k}{2}}(-1)^r\tfrac{k!}{2r!(k-2r)!}\sum_{s=1}^{\frac{l}{2}}(-1)^s\tfrac{l!}{2s!(l-2s)!}(\lambda+\tfrac{1}{k})^{-2r}(\tfrac{1}{k})^{2r}(\lambda+\tfrac{1}{l})^{-2s}(\tfrac{1}{l})^{2s}\Big) \\
= \;& 0
\end{aligned}
$$

This equation has negative coefficient of λ or negative constant in $k \geq 2$ and $l \geq 2$. We also have negative coefficient d_t or constant in the case k and l are odd, k is odd and l is even, k is even and l is odd.

Case (iv): There exists some $w \in W$, $|W| \geq 2$ such that there exist some $K_w \subset M$, $|K_w| \geq 2$ and some $i \in N$ with for each $j \in K_w$, $0 < p_{ij} < 1$, $\sum_{j \in K_w} p_{ij} = 1$ and $q_{ji} = 1$
or there exists some $v \in V$, $|V| \geq 2$ s.t. there exist some $L_v \subset N$, $|L_v| \geq 2$ and some $j \in M$ with for each $i \in L_v$, $0 < q_{ji} < 1$, $\sum_{j \in L_v} q_{ij} = 1$ and $p_{ij} = 1$.

It is clear that Case (iv) necessarily has some negative coefficient d_t or negative constant from the above discussion.

Case (v): There exist some $L \subset N$, $|L| = l \geq 2$ and some $K \subset M$, $|K| =$

$k = l$ such that for each $j \in K$, $0 < p_{ij} < 1$ with $\sum_{j\in K} p_{ij} = 1$ and for each $i \in L$, $0 < q_{ji} < 1$ with $\sum_{i\in L} q_{ji} = 1$.

In the case k and l are even, the characteristic equation of the first-order approximated Jacobian matrix evaluated at the rest point close to this system is given by

$$\begin{aligned} & (\lambda+1)^{(n-1)(2n+2+k)}(\lambda+\tfrac{1}{k})^{(n+1)k} \\ & +\textstyle\sum_{r=1}^{\frac{k}{2}}(-1)^r\frac{k!}{2r!(k-2r)!}(\lambda+1)^{(n-1)(2n+2+k)}(\lambda+\tfrac{1}{k})^{(n+1)k}(\lambda+\tfrac{1}{k})^{-2r}(\tfrac{1}{k})^{2r} \\ = & (\lambda+1)^{(n-1)(2n+2+k)}(\lambda+\tfrac{1}{k})^{(n+1)k} \\ & (1+\textstyle\sum_{r=1}^{\frac{k}{2}}(-1)^r\frac{k!}{2r!(k-2r)!}(\lambda+\tfrac{1}{k})^{-2r}(\tfrac{1}{k})^{2r}) \end{aligned}$$

In the case k is odd, it is given by

$$\begin{aligned} & (\lambda+1)^{(n-1)(2n+2+k)}(\lambda+\tfrac{1}{k})^{(n+1)k} \\ & +\textstyle\sum_{r=1}^{\frac{k-1}{2}}(-1)^r\frac{k!}{2r!(k-2r)!}(\lambda+1)^{(n-1)(2n+2+k)}(\lambda+\tfrac{1}{k})^{(n+1)k}(\lambda+\tfrac{1}{k})^{-2r}(\tfrac{1}{k})^{2r}=0. \end{aligned}$$

These equations has negative coefficient d_t or negative constant in $k \geq 3$.

All Nash strategies that have the above properties necessarily destroy the stability. Moreover, it is clear that all Nash strategies necessarily destroy the stability of the rest point in all the combination of the above properties. □

4 Conclusion

In this note, we investigate the stability of rest points of sender-receiver games under the selection-mutation dynamics when the number of states of the world is not the same as the number of signals. We have focused on neutrally stable strategies in sender-receiver games, an extended-signaling system and a particular hybrid system. These can be defined in the case $n \neq m$, and have Lyapunov stability under the replicator dynamics. On the other hand, all neutrally stable strategies have Lyapunov stability under the selection-mutation dynamics in the case $n = m = 3$.

We have generally proved that a rest point exists close to each extended-signaling system and this point can be asymptotically stable under the selection-mutation dynamics when the number of states of the world is not the same as the number of signals. Restricting to $|m - n| = 1$, we have paticular-hybrid systems. we have also proved that there exists a rest point that can be asymptotically stable under the selection-mutation dynamics. However, though there exisit each rest point the other Nash strategies, these rest points are not asymptotically stable.

Table 1: The first-order approximated Jacobian matrix of the selection-mutation dynamics evaluated at the rest point close to an extended-signaling sysytem (P_4^*, Q_4^*)

$$P_4^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, Q_4^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \text{ where } Z_{P_4^*} = \{3\}.$$

	∂p_{11}	∂p_{12}	∂p_{13}	∂p_{21}	∂p_{22}	∂p_{23}	∂q_{11}	∂q_{12}	∂q_{21}	∂q_{22}	∂q_{31}	∂p_{32}
$\partial \dot{p}_{11}$	$-1+2\varepsilon+\delta$	$-\delta$	$-\frac{1}{2}+\frac{3}{2}\varepsilon$	0	0	0	3ε	0	$-\varepsilon$	0	-2ε	0
$\partial \dot{p}_{12}$	$-\varepsilon$	$-1-\varepsilon+2\delta$	$-\frac{1}{2}\varepsilon$	0	0	0	$-\varepsilon$	0	ε	0	0	0
$\partial \dot{p}_{13}$	-2ε	0	$-\frac{1}{2}-2\varepsilon+\delta$	0	0	0	-2ε	0	0	0	2ε	0
$\partial \dot{p}_{21}$	0	0	0	$-1-\varepsilon+2\delta$	$-\varepsilon$	$-\frac{1}{2}\varepsilon$	0	ε	0	$-\varepsilon$	0	0
$\partial \dot{p}_{22}$	0	0	0	$-\delta$	$-1+2\varepsilon+\delta$	$-\frac{1}{2}+\frac{3}{2}\varepsilon$	0	$-\varepsilon$	0	3ε	0	-2ε
$\partial \dot{p}_{23}$	0	0	0	0	-2ε	$-\frac{1}{2}-2\varepsilon+\delta$	0	0	0	-2ε	0	2ε
$\partial \dot{q}_{11}$	δ	0	0	$-\delta$	0	0	$-1+3\varepsilon$	$-\varepsilon$	0	0	0	0
$\partial \dot{q}_{12}$	$-\delta$	0	0	δ	0	0	$-\delta$	$-1+4\varepsilon-\delta$	0	0	0	0
$\partial \dot{q}_{21}$	0	δ	0	0	$-\delta$	0	0	0	$-1+4\varepsilon-\delta$	$-\delta$	0	0
$\partial \dot{q}_{22}$	0	$-\delta$	0	0	δ	0	0	0	$-\varepsilon$	$-1+3\varepsilon$	0	0
$\partial \dot{q}_{31}$	0	0	$\frac{1}{4}$	0	0	$-\frac{1}{4}$	0	0	0	0	$-\varepsilon-2\delta$	$-\varepsilon$
$\partial \dot{q}_{32}$	0	0	$-\frac{1}{4}$	0	0	$\frac{1}{4}$	0	0	0	0	$-\varepsilon$	$-\varepsilon-2\delta$

Table 2.A: The list of all entries of the first-order approximated Jacobian matrix of the selection-mutation dynamics evaluated at the rest point close to each extended-signaling sysytem (for $n \le m$)

	$s=i,\, t=j$	$s=i, (s,t) \in I_2$	$s=i,\,(s,t)\in I_3$	$s \neq i$		$t=j,\, s=i,$	$t=j,\, s\neq i$	$t\neq j,\, s=i,\,(s,t)\in I_2$	$t=j,\, s\neq i,\,(i,j)\in I_2$	$t\neq j,\, s=i,\,(s,t)\in I_3$	$t=j, s\neq i,\,(i,j)\in I_3$
$\frac{\partial \dot{p}_{ij}}{\partial p_{st}}$; $(i,j)\in I_1$	$-1+\frac{-3n+mn+2}{n-1}\varepsilon+(n-1)\delta$	$-\delta$	$-\frac{1}{n}+\frac{mn-2n+1}{n(n-1)}\varepsilon$	0	$\frac{\partial \dot{p}_{ij}}{\partial q_{ts}}$; $(i,j)\in I_1$	$-\frac{2n-mn-1}{n-1}\varepsilon$	0	$-\varepsilon$	0	$-\frac{n}{n-1}\varepsilon$	0
the number of entries	$n\cdot 1$	$n\cdot(n-1)$	$n\cdot(m-n)$	$n\cdot m(n-1)$		$n\cdot 1$	$n\cdot(n-1)$	$n\cdot(n-1)$	$n\cdot(n-1)^2$	$n\cdot(m-n)$	$n\cdot(n-1)(m-n)$

	$s=i,\,(s,t)\in I_1$	$s=i,\, t=j$	$s=i,\,(s,t)\in I_2$	$s=i,\,(s,t)\in I_3$	$s\neq i$		$t=j,\, s=i$	$t=j,\, s\neq i,\,(i,j)\in I_1$	$t\neq j,\, s=i,\,(s,t)\in I_1$	$t=j,\, s\neq i,\,(s,t)\notin I_1$	$t\neq j,\, s\neq i,\,(s,t)\notin I_1$
$\frac{\partial \dot{p}_{ij}}{\partial p_{st}}$; $(i,j)\in I_2$	$-\varepsilon$	$-1-\varepsilon+n\delta$	0	$-\frac{1}{n}\varepsilon$	0	$\frac{\partial \dot{p}_{ij}}{\partial q_{ts}}$; $(i,j)\in I_2$	ε	0	$-\varepsilon$	0	0
the number of entries	$n(n-1)\cdot 1$	$n(n-1)\cdot 1$	$n(n-1)\cdot(n-2)$	$n(n-1)\cdot(m-n)$	$n(n-1)\cdot m(n-1)$		$n(n-1)\cdot 1$	$n(n-1)\cdot(n-1)$	$n(n-1)\cdot 1$	$n(n-1)\cdot 1$	$n(n-1)\cdot n(m-2)$

	$s=i,\,(s,t)\in I_1$	$s=i,(s,t)\in I_2$	$s=i,\, t=j$	$s=i,\,(s,t)\in I_3$	$s\neq i$		$s=i,\,(s,t)\in I_1$	$s=i,(s,t)\in I_2$	$t=j, s=i$	$s\neq i$
$\frac{\partial \dot{p}_{ij}}{\partial p_{st}}$; $(i,j)\in I_3$	$-\frac{n}{n-1}\varepsilon$	0	$\frac{1}{n}-1-\frac{n}{n-1}\varepsilon+(n-1)\delta$	$-\frac{1}{n-1}\varepsilon$	0	$\frac{\partial \dot{p}_{ij}}{\partial q_{ts}}$; $(i,j)\in I_3$	$-\frac{n}{n-1}\varepsilon$	0	$\frac{n}{n-1}\varepsilon$	0
the number of entries	$n(m-n)\cdot 1$	$n(m-n)\cdot(n-1)$	$n(m-n)\cdot 1$	$n(m-n)\cdot(m-n-1)$	$n(m-n)\cdot m(n-1)$		$n(m-n)\cdot 1$	$n(m-n)\cdot(n-2)$	$n(m-n)\cdot 1$	$n(m-n)\cdot m(n-1)$

	$s=i,\, t=j$	$s=i,\,(s,t)\in I_2$	$s=i,\,(s,t)\in I_3$	$t=j,\, s\neq i,\,(s,t)\in I_2, I_3$	$t\neq j$		$t=j, s=i$	$t=j, s\neq i$	$t\neq j$
$\frac{\partial \dot{q}_{ji}}{\partial p_{st}}$; $(i,j)\in I_1$	δ	0	0	$-\delta$	0	$\frac{\partial \dot{q}_{ji}}{\partial q_{ts}}$; $(i,j)\in I_1$	$-1+\frac{-2n+mn+1}{n-1}\varepsilon+(n-2)\delta$	$-\varepsilon$	0
the number of entries	$n\cdot 1$	$n\cdot(n-1)$	$n\cdot(m-n)$	$n\cdot(n-1)$	$n\cdot(n-1)(m-1)$		$n\cdot 1$	$n\cdot(n-1)$	$n\cdot n(m-1)$

	$s\neq i, t=j,\,(s,t)\in I_1$	$s=i, t\neq j, (i,j)\in I_1$	$s=i,\, t=j$	$s=i,\, t\neq j$	$t\neq j$		$t=j,\, s\neq i,\,(s,t)\in I_1$	$t=j,\, s=i$	$t=j, s\neq i,\,(s,t)\notin I_1$	$t\neq j$
$\frac{\partial \dot{q}_{ji}}{\partial p_{st}}$; $(i,j)\in I_2$	$-\delta$	0	δ	0	0	$\frac{\partial \dot{q}_{ji}}{\partial q_{ts}}$; $(i,j)\in I_2$	$-\delta$	$-1+\frac{mn-n}{n-1}\varepsilon-\delta$	0	0
the number of entries	$n(n-1)\cdot 1$	$n(n-1)\cdot(m-1)$	$n(n-1)\cdot 1$	$n(n-1)\cdot(m-1)$	$n(n-1)\cdot m(m-2)$		$n(n-1)\dot{1}$	$n(n-1)\dot{1}$	$n(n-1)\dot{(}n-2)$	$n(n-1)\dot{n}(m-1)$

	$s=i,\, t=j$	$s=i, t\neq j$	$s\neq i,\, t=j$	$s\neq i,\, t\neq j$		$t=j, s\neq i$	$t=j,\, s=i$	$t\neq j$
$\frac{\partial \dot{q}_{ji}}{\partial p_{st}}$; $(i,j)\in I_3$	$\frac{1}{n}-\frac{1}{n^2}$	0	$-\frac{1}{n^2}$	0	$\frac{\partial \dot{q}_{ji}}{\partial q_{ts}}$; $(i,j)\in I_3$	$-\frac{1}{n-1}\varepsilon$	$-\frac{1}{n-1}\varepsilon-n\delta$	0
the number of entries	$n(m-n)\cdot 1$	$n(m-n)\cdot(m-1)$	$n(m-n)\cdot(n-1)$	$n(m-n)\cdot(n-1)(m-1)$		$n(m-n)\cdot(n-1)$	$n(m-n)\cdot 1$	$n(m-n)\cdot n(m-1)$

Table 2.B: The list of all entries of the first-order approximated Jacobian matrix of the selection-mutation dynamics evaluated at the rest point close to each extended-signaling sysytem (for $n \geq m$)

	$s=i, t=j$	$s=i, t\neq j$	$s\neq i$		$t=j,\ s=i$	$t=j,\ (s,t)\in I_4$	$t=j,\ (s,t)\in I_5$	$s=i,\ t\neq j,\ (s,t)\in I_4, I_5$	$s\neq i$
$\frac{\partial \dot{p}_{ij}}{\partial p_{st}};\ (i,j)\in I_1$	$-1+(m-2)\varepsilon+\frac{-2m+mn-1}{n-1}\delta$	$-\delta$	0	$\frac{\partial \dot{p}_{ij}}{\partial q_{ts}};\ (i,j)\in I_1$	ε	0	0	$-\varepsilon$	0
the number of entries	$m\cdot 1$	$m\cdot(m-1)$	$m\cdot m(n-1)$		$m\cdot 1$	$m\cdot(m-1)$	$m\cdot(n-m)$	$m\cdot(m-1)$	$m\cdot(n-1)(m-1)$

	$s=i,\ t\neq j,\ (s,t)\in I_1$	$s=i, t=j$	$s=i, t\neq j,\ (s,t)\notin I_1$	$s\neq i$		$t\neq j, s=i,\ (s,t)\in I_1$	$t=j, s\neq i, (i,j)\in I_1$	$t=j,\ s=i$	$t=j,\ s\neq i$	$s\neq i$
$\frac{\partial \dot{p}_{ij}}{\partial p_{st}};\ (i,j)\in I_4$	$-\varepsilon$	$-1+\frac{mn-m}{m-1}\delta$	0	0	$\frac{\partial \dot{p}_{ij}}{\partial q_{ts}};\ (i,j)\in I_4$	$-\varepsilon$	0	ε	0	0
the number of entries	$m(m-1)\cdot 1$	$m(m-1)\cdot 1$	$m(m-1)\cdot(m-2)$	$m(m-1)\cdot m(n-1)$		$m(m-1)\cdot 1$	$m(m-1)\cdot(n-1)$	$m(m-1)\cdot 1$	$m(m-1)\cdot(n-1)$	$n(m-1)\cdot m(n-2)$

	$s=i, t\neq j$	$s=i,\ t=j$	$s\neq i$		$t=j,\ s=i$	$t=j, s\neq i$	$t\neq j,\ s=i$	$t\neq j,\ s\neq i$
$\frac{\partial \dot{p}_{ij}}{\partial p_{st}};\ (i,j)\in I_5$	$-\frac{1}{m-1}\delta$	$-m\varepsilon-\frac{1}{m-1}\delta$	0	$\frac{\partial \dot{p}_{ij}}{\partial q_{ts}};\ (i,j)\in I_5$	$\frac{1}{m}-\frac{1}{m^2}$	0	$-\frac{1}{m^2}$	0
the number of entries	$m(n-m)\cdot 1$	$m(n-m)\cdot(m-1)$	$m(n-m)\cdot m(n-1)$		$m(n-m)\cdot 1$	$m(n-m)\cdot(n-1)$	$m(n-m)\cdot(m-1)$	$m(n-m)\cdot(n-1)(m-1)$

	$s=i,\ t=j,$	$s=i, t\neq j$	$s\neq i,\ t=j,\ (s,t)\in I_4$	$s=i,\ t\neq j,\ (i,j)\in I_4$	$s\neq i,\ t=j,\ (s,t)\in I_5$	$s=i, t\neq i,\ (i,j)\in I_5$		$t=j,\ s=i$	$t=j, (s,t)\in I_4$	$t=j,\ (s,t)\in I_5$	$t\neq j$
$\frac{\partial \dot{q}_{ji}}{\partial p_{st}};\ (i,j)\in I_1$	$-\frac{2m-mn-1}{m-1}\delta$	0	$-\delta$	0	$-\frac{m}{m-1}\delta$	0	$\frac{\partial \dot{q}_{ji}}{\partial q_{ts}};\ (i,j)\in I_1$	$-1+(m-1)\varepsilon+\frac{-3m+mn+2}{m-1}\delta$	$-\varepsilon$	$-\frac{1}{m}+\frac{mn-2m+1}{m(m-1)}\delta$	0
the number of entries	$m\cdot 1$	$m\cdot(m-1)$	$m\cdot(m-1)$	$m\cdot(m-1)^2$	$m\cdot(n-m)$	$m\cdot(m-1)(n-m)$		$m\cdot 1$	$m\cdot(m-1)$	$m\cdot(n-m)$	$m\cdot n(m-1)$

	$s=i,\ t=j$	$s=i,\ t\neq j,\ (i,j)\in I_1$	$s\neq i,\ t=j,\ (s,t)\in I_1$	$s=i,\ t\neq j,\ (s,t)\notin I_1$	$s\neq i,\ t\neq j,\ (s,t)\notin I_1$		$t=j,\ (s,t)\in I_1$	$t=j,\ s=i$	$t=j,\ (s,t)\in I_4$	$t=j,\ (s,t)\in I_5$	$t\neq j$
$\frac{\partial \dot{q}_{ji}}{\partial p_{st}};\ (i,j)\in I_4$	δ	0	$-\delta$	0	0	$\frac{\partial \dot{q}_{ji}}{\partial q_{ts}};\ (i,j)\in I_4$	$-\delta$	$-1+m\varepsilon-\delta$	0	$-\frac{1}{m}\delta$	0
the number of entries	$m(m-1)\cdot 1$	$m(m-1)\cdot(m-1)$	$m(m-1)\cdot 1$	$m(m-1)\cdot(m-1)$	$m(m-1)\cdot m(n-2)$		$m(m-1)\cdot 1$	$m(m-1)\cdot 1$	$m(m-1)\cdot(m-2)$	$m(m-1)\cdot(n-m)$	$m(m-1)\cdot n(m-1)$

	$t=j,\ (s,t)\in I_1$	$t=j, (s,t)\in I_4$	$s=i, t=j$	$t\neq j$		$t=j,\ (s,t)\in I_1$	$t=j, (s,t)\in I_4$	$t=j,\ s=i$	$t=j,\ (s,t)\in I_5$	$t\neq j$
$\frac{\partial \dot{q}_{ji}}{\partial p_{st}};\ (i,j)\in I_5$	$-\frac{m}{m-1}\delta$	0	$\frac{m}{m-1}\delta$	0	$\frac{\partial \dot{q}_{ji}}{\partial q_{ts}};\ (i,j)\in I_5$	$-\frac{m}{m-1}\delta$	0	$\frac{1}{m}-1+(m-1)\varepsilon-\frac{m}{m-1}\delta$	$-\frac{1}{m-1}\delta$	0
the number of entries	$m(n-m)\cdot 1$	$m(n-m)\cdot(m-2)$	$m(n-m)\cdot 1$	$m(n-m)\cdot n(m-1)$		$m(n-m)\cdot 1$	$m(n-m)\cdot(m-1)$	$m(n-m)\cdot 1$	$m(n-m)\cdot(n-m-1)$	$m(n-m)\cdot n(m-1)$

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