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Abstract

We estimate a small-scale macroeconomic model for Japan by taking into account the nonlinearity stemming from the zero lower bound (ZLB) of the nominal interest rate. To this end, we apply the Sequential Monte Carlo Squared method to the case of Japan, where the ZLB has constrained the country's monetary policy for a considerably long period. Employing a nonlinear estimation is crucial to deriving implications for monetary policy. For example, the Bayesian model selection suggests that past experience of recessions reducing the nominal interest rate to zero is carried over to today's monetary policy. However, a nonlinear estimation has little effect on the estimate of the natural rate of interest, which has often been negative since the mid-1990s.

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1 Introduction

In modern macroeconomic analyses, dynamic stochastic general equilibrium (DSGE) models are a workhorse tool for assessing the state of the economy and the effects of policy. However, the effective zero lower bound (ELB or ZLB; hereafter, we refer to the ZLB) on the nominal interest rate poses a serious technical challenge. The nonlinearity produced by the ZLB means that solving a rational expectation equilibrium is computationally difficult. Estimating the model is even more difficult, because it requires that we repeat the process of solving the rational expectation equilibrium for a number of parameter values. Neglecting the ZLB may generate biased estimates and, thus, incorrect policy implications, because the ZLB often changes the quantitative results, as pointed by Fernández-Villaverde et al. (2015), Boneva, Braun and Waki (2016), and Nakata (2017). The problem of the ZLB is particularly serious in the case of Japan because the Japanese economy has been trapped at the effective ZLB since 1995.

In this study, we attempt to estimate a DSGE model for Japan, where the model is nonlinear and stochastic. To facilitate the computations, we use one of the simplest New Keynesian models that abstracts many of the important features of standard DSGE models, such as consumption habit, capital formation, and wage stickiness. However, we incorporate nonlinearity and stochastic shocks, which make the ZLB occasionally binding.

We use an estimation to investigate two main points. First, we analyze Japan's monetary policy. Although many central banks have conducted commitment (forward guidance) policies under the ZLB in the aftermath of financial crises, estimating such a policy is difficult because we have to embed the ZLB explicitly in a model. Specifically, we address the question of which past variable a central bank refers to when its monetary policy has an inertia. In some cases, the monetary policy rule refers to the *actual* nominal interest rate in the previous period, which takes a value greater than or equal to zero. In others, it refers to the *notional* interest rate in the previous period, which can take values lower than zero. Ceteris paribus, the latter type of monetary policy rule is considered to have a larger effect on the economy at the ZLB, because today's interest rate is likely to be lower as a result of a negative notional interest rate.[1] Therefore, to assess the effects of a monetary policy, we need to identify a true monetary policy rule, and then estimate its parameters by fully taking into account the nonlinearity stemming from the ZLB.[2] However, most empirical studies neglect

[1]Reifschneider and Williams (2000) propose a monetary policy under the ZLB similar to the second type of monetary policy rule.

[2]A partial-equilibrium approach may enable us to identify a true monetary policy rule by estimating the response of nominal interest rates to economic disturbances. In contrast, our estimation based on a DSGE model takes into account not only the response of nominal interest rates to economic disturbances but also the effects of the monetary policy on the economy. Thus, if the forward guidance puzzle is significant, that is, if the theoretical power of forward guidance is too strong (Del Negro, Giannoni, and Patterson (2015) and McKay, Nakamura, and Steinsson (2016)), then our estimation may support the first type of monetary

the ZLB, and those that do take it into account derive models that rest on one of the two monetary policy rules. For example, Aruoba, Cuba-Borda, and Schorfheide (2017) assume the first type identified above, while Gust et al. (2017) and Richter and Throckmorton (2016) assume the second type. In addition, we investigate impulse response functions to shocks given various kinds of monetary policy specifications as well as the probability that the ZLB constrains monetary policy.

The second focus of this study is the natural rate of interest. The natural rate of interest is the real interest rate that would lead to price stability (Wicksell (1936)). It plays a pivotal role in New Keynesian models, such as those of Woodford (2003). For example, when the actual real interest rate exceeds the natural rate of interest, real economic activity is dampened and the inflation rate decreases. Krugman (1998) points out the possibility that the equilibrium real interest rate in Japan decreased, although he does not use the term natural rate or estimate it quantitatively. By estimating the natural rate of interest, we seek to explain Japan's stagnant recessions, the so-called lost decades, since the early 1990s.

To estimate the model that includes the ZLB, we adopt two approaches recently introduced in the literature. First, we use the method the *time iteration with linear interpolation* (TL) method recommended by Richter, Throckmorton, and Walker (2014) to solve for the rational expectation equilibrium. This method is within the class of policy function iterations and, according to the authors, is flexible, accurate, and speedy. Second, we estimate the parameters using the *Sequential Monte Carlo Squared* (SMC^2) method, developed by Chopin, Jacob, and Papaspiliopoulos (2013), and applied to DSGE models by Herbst and Schorfheide (2015). In this method, we evaluate the likelihood of a nonlinear model, given a certain parameter set, by generating particles of endogenous variables (often called a particle filter). Further, by sampling the particles of the parameter sets, we derive the posterior distribution of the parameters (the Sequential Monte Carlo or SMC method).[3] As Chopin, Jacob, and Papaspiliopoulos (2013), Herbst and Schorfheide (2015), and Fernández-Villaverde, Rubio-Ramírez, and Schorfheide (2016) argue, compared with the particle Markov chain Monte Carlo (MCMC) technique, the SMC^2 method leads to a more reliable posterior inference and, thus, we do not need large measurement errors.

Our main findings are as follows. First, we show that a nonlinear estimation is crucial to deriving implications for monetary policy. For example, the Bayesian model selection chooses a model in which today's monetary policy depends on the notional interest rate,

policy rule rather than the second type.

[3]Before the development of the SMC^2 method, using the SMC method for parameter estimation was limited to linear state-space models, where the Kalman filter can be applied to evaluate the likelihood. See, for example, Chopin (2002) and Herbst and Schorfheide (2014). Alternatively, when estimating nonlinear state-space models, past studies use a particle filter by combining it with the Markov chain Monte Carlo (MCMC) technique (e.g., the Metropolis–Hastings (MH) algorithm), developed by Andrieu, Doucet, and Holenstein (2010), and often called the particle MCMC technique. See also Kitagawa (1996) and Fernandez-Villaverde and Rubio-Ramirez (2005) for the particle filter.

which can take negative values in the previous period. This suggests that past experience of recessions in reducing the nominal interest rate to zero is carried over to today's monetary policy, preventing the Bank of Japan from tightening its monetary policy, even if the economy improves. This type of carry-over policy is interpreted as a forward guidance (commitment) policy. That is, the Bank of Japan has been implementing forward guidance policy at the ZLB by committing to continuing the zero-rate policy for a long time. Impulse response functions to a monetary policy shock vary significantly, depending on the model for the monetary policy rule and on the sign of the shock.

Second, to our surprise, we find that the ZLB does not produce a bias for the estimated natural rate of interest. Although the ZLB produces considerable biases for parameter estimates, which affects policy implications, the estimated natural rate of interest shows little variation between models with different monetary policy specifications. Even if a misspecification produces biased parameters, their effect on the natural rate of interest is canceled out by the effect of biased shocks. The natural rate of interest has often been negative since the mid-1990s, mainly as a result of weak demand shocks.

This study is not the first attempt to estimate a nonlinear DSGE model with the ZLB.[4] The two studies closest to ours are those of Gust et al. (2017) and Richter and Throckmorton (2016). However, there are three main differences between these works and ours. First, they use data for the United States, where the ZLB is only relevant for a few years, whereas the nominal interest rate has been almost zero for two decades in Japan. Thus, the constrained linear model, which is linear except for the ZLB, performs very poorly for Japan, in contrast to the findings of Richter and Throckmorton (2016). Second, we use the SMC^2 method to generate particles for both the model variables (shock processes) and the model parameters. In contrast, the two aforementioned studies use the MCMC technique to estimate the model. As in the previous two works, we generate particles for endogenous variables in order to compute the likelihood, because the model is nonlinear and, thus, we cannot employ the Kalman filter. In addition, in the SMC^2 case, we derive the posterior distribution of the parameters by sampling the particles of the parameter sets. As we later explain in detail, the SMC^2 method enables us to obtain a reliable posterior inference. We do not need to assume large measurement errors for feasibility, unlike Gust et al. (2017) and Richter and Throckmorton (2016). Third, Gust et al. (2017) estimate a far richer medium-sized DSGE model than that of Richter and Throckmorton (2016) or the model used in this paper.

Studies on developments in the natural rate of interest include the works of Krugman (1998), Laubach and Williams (2003, 2016), Neiss and Nelson (2003), Andrés, López-Salido,

[4]Hirose and Sunakawa (2015), Hirose and Inoue (2016), and Aruoba, Cuba-Borda, and Schorfheide (2017) estimate a model without the ZLB, although they generate data and conduct simulations that incorporate the ZLB. By assuming that the duration of ZLB, τ_t, is foreseen perfectly in each period t, Kulish, Morley, and Robinson (2017) estimate τ_t and time-varying policy functions given estimated τ_t. Aoki and Ueno (2012) and Kim and Pruitt (2017) use forward rate curves and forecasters' surveys, respectively.

and Nelson (2009), Hall (2011), Barsky, Justiniano, and Melosi (2014), Ikeda and Saito (2014), Cúrdia (2015), Cúrdia et al. (2015), Bank of Japan (2016), Del Negro et al. (2017), Hirose and Sunakawa (2017), and Holston, Laubach, and Williams (2017) among others. With the exception of the recent work of Hirose and Sunakawa (2017), none of these studies use the DSGE model while explicitly considering the ZLB. Hirose and Sunakawa (2017) evaluate the natural rate of interest using the DSGE model with the ZLB for the United States, but do not estimate the model with the ZLB. Instead, they estimate the model without the ZLB for the periods before the ZLB constrains the economy and then evaluate the natural rate of interest using the estimated parameters for the extended periods.

The remainder of this paper is structured as follows. Section 2 briefly explains our model, and Section 3 outlines our estimation methods. Sections 4 and 5 discuss our estimation results with regard to monetary policy and the natural rate of interest, respectively. Lastly, Section 6 concludes.

2 Model

Our model is one of the simplest New Keynesian models. The economy consists of a representative household, firms, and a central bank. Firms consist of monopolistically competitive intermediate-good producers and perfectly competitive final-good producers. We consider three types of monetary policy, including one that abstracts the ZLB. The economy is subject to three types of exogenous shocks: a discount factor (preference), technology, and monetary policy.

2.1 Household

A representative household maximizes its welfare as follows:

$$E_t\left[\sum_{j=0}^{\infty}\beta^j Z_{t+j}^b\left\{\frac{C_{t+j}^{1-\sigma}}{1-\sigma}-\frac{(A_{t+j})^{1-\sigma}\chi l_{t+j}^{1+\omega}}{1+\omega}\right\}\right], \tag{1}$$

subject to the budget constraint $C_t + B_t/P_t \leq W_t l_t + R_{t-1}B_{t-1}/P_t + T_t$, where C_t, l_t, P_t, W_t, R_t, and T_t represent consumption, labor services, the aggregate price level, the real wage, the nominal rate of return, and the lump-sum transfer in period t, respectively. In addition, B_t represents the holding of one-period riskless bonds at the end of period t. Parameter $\beta \in (0,1)$ is the subjective discount factor, $\sigma > 0$ measures the inverse of the intertemporal elasticity of substitution of consumption, $\omega > 0$ is the inverse of the labor supply elasticity, and $\chi > 0$ is the scale factor. Finally, Z_t^b represents a stochastic shock to the discount factor (preference), with a unit mean and obeying the following AR(1) process:

$$\log(Z_t^b) = \rho^b \log(Z_{t-1}^b) + \epsilon_t^b. \tag{2}$$

As in Erceg, Guerrieri, and Gust (2006), we allow for preferences over leisure to shift with the level of technology, A_t, to ensure the existence of the balanced growth path.

2.2 Firms

The final-good firm faces perfect competition and produces output Y_t by choosing the combination of intermediate inputs $Y_{f,t}$ ($f \in [0,1]$) that maximizes its profit, subject to the Dixit-Stiglitz form of aggregations $Y_t = \left\{ \int_0^1 Y_{f,t}^{\frac{\varepsilon-1}{\varepsilon}} df \right\}^{\frac{\varepsilon}{\varepsilon-1}}$, where $\varepsilon > 1$ represents the elasticity of substitution between intermediate goods.

Intermediate-good firm f produces output $Y_{f,t}$ using the production technology given by $Y_{f,t} = A_t l_{f,t}$. The technology shock A_t obeys the I(1) process with a non-zero growth rate of γ^a, where $\mu_t^a \equiv \log(A_t/A_{t-1}) - \gamma^a$ is given by

$$\mu_t^a = \rho^a \mu_{t-1}^a + \epsilon_t^a. \tag{3}$$

Intermediate-good firm f maximizes its firm value by setting the optimal price $P_{f,t}$ in period t in the presence of a Rotemberg-type price adjustment cost:

$$E_t \left[\sum_{j=0}^{\infty} \beta^j \frac{\Lambda_{t+j} Z_{t+j}^b}{\Lambda_t Z_t^b} \left(\frac{P_{f,t+j}}{P_{t+j}} - \frac{W_{t+j}}{A_{t+j}} - \frac{\phi}{2} \left(\frac{P_{f,t+j}}{P_{f,t+j-1}} - \pi^* \right)^2 \right) Y_{f,t+j} \right] \tag{4}$$

subject to downward-sloping demand, where Λ_t and π^* represent the stochastic discount factor and the target inflation rate, respectively, and ϕ captures the degree of the Rotemberg-type price adjustment cost.

2.3 Central Bank

In this study, we consider three types of monetary policy models. Models 1 and 2 incorporate the ZLB as

$$R_t = \max\left(1, R_t^*\right), \tag{5}$$

where R_t^* represents the notional interest rate, which can take values less than one. The actual interest rate R_t cannot be less than one. The third model, Model without the ZLB, ignores the ZLB constraint. Models 1 and 2 are characterized by the monetary policy rules,

$$R_t^* = \left(R_{t-1}^*\right)^{\rho^r} \left(r^* \pi^* \left(\frac{\pi_t}{\pi^*} \right)^{\psi_\pi} \left(\frac{Y_t/A_t}{Y_t^*/A_t} \right)^{\psi_y} \right)^{1-\rho^r} e^{\epsilon_t^r}, \tag{6}$$

and

$$R_t^* = \left(R_{t-1}\right)^{\rho^r} \left(r^* \pi^* \left(\frac{\pi_t}{\pi^*} \right)^{\psi_\pi} \left(\frac{Y_t/A_t}{Y_t^*/A_t} \right)^{\psi_y} \right)^{1-\rho^r} e^{\epsilon_t^r}, \tag{7}$$

respectively, where ρ^r, ψ_π, and ψ_y capture the monetary policy responses to the past interest rate, the inflation rate, and the output gap, respectively. Models 1 and 2 differ only in which interest rate the central bank refers to. In Model 1, this is the notional interest rate, and in Model 2 is the actual interest rate. We denote the steady-state natural rate of interest, the natural level of output, and the inflation rate by r^*, Y_t^*, and π_t, respectively, where ϵ_t^r represents a stochastic *i.i.d.* shock to the monetary policy with zero mean.

Compared with Model 2, Model 1 incorporates a stronger commitment for future policy. Because R_t^* can be below zero and depends on R_{t-1}^*, the experiences of adverse shocks in the past limit the actions of the central bank for long periods. In other words, the central bank compensates for its inability to lower the policy rate below zero by continuing the zero-rate policy, ceteris paribus.

2.4 Closing the Model

The goods market is cleared as

$$Y_t = C_t + \phi \left(\pi_t - \pi^*\right)^2 Y_t/2. \tag{8}$$

In addition, the flexible-price equilibrium is defined as that without the Rotemberg-type price adjustment cost.

The natural rate of interest r_t^* in the model is equal to the real rate of return in this flexible-price economy. Similarly, the natural level of output Y_t^* is equal to the output in the economy with flexible prices.

3 Methodology

In this section, we outline how we solve and estimate the nonlinear DSGE model with the ZLB. We then explain our data and the prior specifications, including the size of the measurement errors. Finally, we discuss the advantage of the SMC^2 method.[5]

3.1 Model Solution

We solve the rational expectation equilibrium of our model using the *time iteration method with linear interpolation* (TL) method. This is within the class of policy function iteration methods, with Richter, Throckmorton, and Walker (2014) showing that it provides the best balance between speed and accuracy.

More precisely, we solve for the rational expectation equilibrium or policy function, given the parameter set θ. Note that, in our model, the policy function of any variable X_t is

[5]See the Appendix for details. In Appendix A, we explain the method to solve the rational expectation equilibrium. In Appendix B, we explain the method to estimate our model.

expressed as $X_t = X(\mu_t^a, Z_t^b, \epsilon_t^r, R_{t-1}^*)$, because there are three shocks and one state variable, R_{t-1}^*. Intuitively, the TL method begins with a time iteration for a policy function until the intertemporal equations that describe relations between X_t and $E_t(X_{t+1})$ are satisfied at every node. Compared with the fixed-point iteration, it is costly to call a nonlinear solver on each node. However, the method is more stable, precisely because the policy function is optimized on each node. We then locally approximate the policy functions using linear interpolation. Compared with global approximation methods, such as the projection method using the Chebyshev polynomial basis, linear interpolation is considered to perform better in an environment where the ZLB produces kinks in the policy functions.

The rational expectation equilibrium obtained using the TL method is not the deflationary equilibrium shown by Benhabib, Schmitt-Grohe, and Uribe (2001). The equilibrium derived using the TL method is a minimum state variable solution, which, as McCallum (1999) discusses, rules out an indeterminate equilibrium, such as a deflationary equilibrium. See Aruoba, Cuba-Borda, and Schorfheide (2017) for an attempt to estimate such an equilibrium.

3.2 Estimation

We estimate the nonlinear DSGE model with the ZLB using a Bayesian technique. We estimate the parameters using the SMC^2 method developed by Chopin, Jacob, and Papaspiliopoulos (2013) and Herbst and Schorfheide (2015). The method comprises the following four steps. In Step 1 (initialization), we draw N_θ particles for parameters θ. We then repeat Steps 2 to 4 below for N_ϕ stages. In Step 2 (correction), given θ, we compute the likelihood $\hat{p}(\mathbf{Y_t}|\theta)$ and the weight $\tilde{W}$. In Step 3 (selection), we resample θ and w based on θ in the previous stage and $\tilde{W}$ in the previous step. Then, in Step 4 (mutation), we propagate θ and $\tilde{W}$ using the MH algorithm.

In Step 2, we solve the model, given θ using the TL method. Then, after drawing N_S particles for shock processes $(\mu_t^a, Z_t^b, \epsilon_t^r)$, we generate the paths of variables $\hat{\mathbf{Y}}_\mathbf{t}$, compare them with observed variables $\mathbf{Y_t}$, and compute the likelihood $\hat{p}(\mathbf{Y_t}|\theta)$ by assuming the presence of the measurement error of $\mathbf{Y_t}$. Note that because the model is nonlinear, we cannot apply the Kalman filter. Thus, we use the particle filter, where we replace $p(\mathbf{Y_t}|\theta)$ by $\hat{p}(\mathbf{Y_t}|\theta)$ using a sufficiently large number of particles N_S with respect to shocks.

In our benchmark estimation, we use the particles of $N_S = 40,000$ and $N_\theta = 1,200$, and the stages of $N_\phi = 10$. For the number of particles of the shock processes, N_S, we follow Richter and Throckmorton (2016). To the best of our knowledge, no studies have determined a criterion for the number of stages N_ϕ, or the number of parameter particles N_θ.[6] However, it is worth noting that, in the SMC^2 method, particles for parameters are

[6]More precisely, early studies, such as those of Herbst and Schorfheide (2014, 2015), may provide a clue as to their choice. However, because we use the likelihood tempering method for the importance sampling,

uncorrelated between each stage. In contrast, in the MCMC technique, a set of sampled parameters (such as median and mode) are highly correlated between each stage (iteration) and, thus, the sampling efficiency is low. This difference means that we cannot apply a convergence test, as we do for the MCMC technique. Furthermore, we do not need a large number of stages. In contrast, Gust et al. (2017) use the MCMC with 40,000 iterations. In our estimation, we choose a relatively large number of parameter particles $N_\phi = 1,200$, while keeping the number of stages low $N_S = 10$. A single estimation takes about a week using a 32-core (Intel Xeon E5-2698v3) computer.

3.3 Data

We use data for Japan from 1983:2Q to 2016:2Q. The beginning period is chosen to coincide with that of the output gap data, which we use for a robustness analysis. In the benchmark estimation, we use $\mathbf{Y_t} = \{\Delta\log Y_t,\ \pi_t,\ R_t\}$: the real per-capita GDP growth rate, the CPI inflation rate, and the overnight call rate. Figure 1 shows the time-series changes in these variables. In obtaining $\Delta\log Y_t$, we divide the real GDP by the population aged 15 years old or over. For π_t, we exclude the effects of consumption tax changes using X12ARIMA. These two variables are quarterly changes from the previous quarter, so R_t is divided by four to make it quarterly. As an alternative to $\Delta\log Y_t$, we later use the output gap $(\log(Y_t/Y_t^*))$ constructed by the Bank of Japan.

3.4 Prior Specifications

We choose parameter values based on Smets and Wouters (2007) and Sugo and Ueda (2008). We fix some of the parameters as $\beta = 0.995$, $\chi = 1$, and $\varepsilon = 6$. Table 1 shows the prior distribution of parameters, where κ is defined by $(\varepsilon - 1)(\omega + \sigma)/(\phi\pi^*)$. Note that, to enhance readability, we express parameters γ^a and $\pi^*$by $100\gamma^a$ and $100(\pi^* - 1)$, respectively. We also discuss the natural rate of interest r_t^* by deducting the value one.

For the measurement errors of $\Delta\log Y_t$, π_t, and R_t, we assume that their sizes are 0.5%, 0.5%, and 0.25% of their actual standard deviations, respectively. Their sizes are far smaller than those in Gust et al. (2017) and Richter and Throckmorton (2016), where they are $\sqrt{0.25} \sim 50\%$ and $\sqrt{0.1} \sim 30\%$, respectively. We assume that the measurement error of R_t is lower than those of $\Delta\log Y_t$ and π_t, but this difference is minor and has little effect on the following results.

whereas they use the data tempering method, our estimation is considered to be less subject to the number of stages.

3.5 Advantage of the SMC^2 over the MCMC

We use the SMC^2 method by generating particles for the shock processes $(\mu_t^a, Z_t^b, \epsilon_t^r)$ and for the parameter set θ, using N_S and N_θ, respectively. In contrast, Gust et al. (2017) and Richter and Throckmorton (2016) use the MCMC technique to estimate the model by generating particles for shock processes only. As Chopin, Jacob, and Papaspiliopoulos (2013), Herbst and Schorfheide (2015), and Fernández-Villaverde, Rubio-Ramírez, and Schorfheide (2016) argue, the SMC^2 method can lead to a more reliable posterior inference than that of the particle MCMC technique. Furthermore, we do not need large measurement errors for the observable variables, unlike Gust et al. (2017) and Richter and Throckmorton (2016). In addition, the SMC^2 method can be paralleled easily.

To understand these advantages better, we use our estimation result (discussed in the next section) as an illustrative example. Figure 2 shows a scatter plot where each dot represents a particle for the value of parameter σ (horizontal axis) and its posterior likelihood (vertical axis). The dots are dense around $\sigma = 1.4$. Their median lies around this level, as shown in the big filled circle in red. Interestingly, the likelihood becomes largest when σ is around 1.5, as shown in the big plus at the top of the figure. However, this circle seems to be an outlier, for three reasons: particles are sparse around this circle; the likelihood drops when σ deviates slightly from the value; and the computation of the likelihood is subject to errors. The last point arises because the model is nonlinear and, thus, the Kalman filter cannot be applied. Therefore, we need to use particles for shock processes and introduce measurement errors for observable variables to approximate the likelihood. Figure 2 shows that our estimation is not trapped at this outlier.

What will happen if we use the MCMC technique in this example? In the MCMC, we compare only two parameter candidates, for example, new candidate σ_1 and previously selected σ_0. Thus, once the aforementioned plus is selected as either σ_1 or σ_0, our estimation tends to be trapped at this point, because the likelihood at its neighborhood is discontinuously lower. In other words, the acceptance probability of the new parameter values is very low. This problem becomes more serious when the measurement errors for the observable variables are smaller, because the likelihood becomes more sensitive to parameter changes. Therefore, we need to assume large measurement errors to keep the acceptance probability of 25% when using the MCMC. For the same reason, the MCMC is considered to be sensitive to the shape of the distribution. If the distribution is close to being bimodal, the posterior inference becomes unstable between the two.[7]

The SMC^2 method can resolve this problem. Because it uses more than two particles for parameter candidates, and allocates weight $\tilde{W}$ to each particle corresponding to its likelihood, particles are much less likely to be stuck at the outlier. The posterior distribution becomes

[7] Herbst and Schorfheide (2015) report that they cannot estimate a Smets and Wouters-type (2007) medium-scale DSGE model using the MCMC technique with the particle filter.

diverse, as shown in the figure.[8]

4 Evaluating Japan's Monetary Policy during the Low Inflation Period

4.1 Model Selection and Parameter Estimates

In this section, we discuss Japan's monetary policy. We first compare the performance of Model 1, Model 2, and Model without the ZLB using a Bayesian estimation by taking into account the ZLB and using data at the ZLB. Because of the general equilibrium, our estimation of the monetary policy rules takes into account not only the response of nominal interest rates to economic disturbances, but also the effects of the monetary policy on the macroeconomy.

Table 2 shows the parameter estimates and marginal likelihood for the three types of models with regard to monetary policy: Model 1, Model 2, and Model without the ZLB. The marginal likelihood is the highest for Model 1, supporting Model 1 with the posterior probability of one over Model 2 and Model without the ZLB. Because the nominal interest rate has been almost zero for the past two decades, the fit of Model without the ZLB is the worst of the three.

This suggests that the past experience of recessions reducing the nominal interest rate to zero prevents the Bank of Japan from tightening its monetary policy today, even if the economy improves. In other words, the Bank of Japan has been conducting a forward guidance policy at the ZLB by committing to continuing the zero-rate policy.

In the following discussions, we report the results mainly for Model 1, because the marginal likelihood is the highest. Hereafter, we call Model 1 the baseline model. The estimates of the structural parameters, such as σ and ω, are within the range reported in earlier studies. The inflation target $\pi^* - 1$ is 0.36% quarterly (i.e., 1.44% annually). This value lies between the Bank of Japan's formal target, 0.5% quarterly, and the mean of the actual inflation rates in the sample period, 0.11% quarterly. The trend component of the technology shock A_t, γ^a, is -0.028% quarterly, which renders a steady-state natural rate of interest $r^* - 1 = e^{\sigma\gamma^a}/\beta - 1$ of 0.46% quarterly.

By comparing the parameter estimates between Model 1 and Model without the ZLB, we observe a significant difference in the inverse of the intertemporal elasticity of substitution of consumption σ, the trend component of the technology shock γ^a, the inflation target π^*,

[8] The likelihood tempering approach in the SMC^2 method is another reason why the SMC^2 performs well. In the mutation step at each stage, a scaling factor is revised to keep the acceptance probability around 25%. This enables us to obtain smooth posterior distribution of parameters around modes, as the number of stages increases.

and the inertia of the monetary policy rule ρ^r. For these four parameters, Model without the ZLB yields smaller values than those of Model 1. In particular, smaller γ^a and π^* suggest that the steady-state nominal interest rate is lower, because it is equal to $e^{\sigma\gamma^a}/\beta - 1 + \pi^*$. It is -0.38% in Model without the ZLB, and is 0.82% in Model 1. Model without the ZLB seems to require a low and negative steady-state nominal interest rate in order to explain the prolonged zero interest rate that has occurred in Japan.

4.2 The Notional Interest Rate

Figure 3 shows developments in the notional nominal interest rate $R_t^* - 1$ in Models 1 and 2, whereas it coincides with the actual nominal interest rate $R_t - 1$ in Model without the ZLB. In Model 1, it has been around -4% annually since 1995. This contributes to reducing future interest rates. On the other hand, the notional nominal interest rate is around -2% annually in Model 2, but this does not constrain the future monetary policy.[9]

Some papers in the finance literature estimate a so-called shadow rate.[10] Like the notional interest rate in our model, the shadow rate can take a negative value, while the actual interest rate is non-negative. However, the shadow rate is estimated very differently. These papers focus on the shape of the yield curve only, using a partial-equilibrium approach. They estimate the shadow rate by extending the Gaussian affine term structure model or the Black (1995) model by incorporating the ZLB. Thus, it is not directly comparable, but it is still worth comparing the notional interest rate in our model with the shadow rate in their models. Here, we borrow the shadow rate reported in Ueno (2017). From Figure 4, we find that the lowest level of the shadow rate is almost the same as that in the notional interest rate in Model 1, which is around -1% quarterly. However, the shadow rate decreased more gradually in the late 1990s than the notional interest rate did. In other words, from the perspective of the yield curve (shadow rate), the Bank of Japan seems to have been easing its monetary policy gradually since the mid-1990s. However, judging from the overall effects on inflation and output, there seems hardly any change in the monetary policy stance since the mid-1990s.

4.3 Impulse Response Functions

A nonlinear estimation is crucial to deriving implications for monetary policy. To show this, we calculate impulse response functions (IRFs) to a monetary policy shock and demonstrate that the IRFs are very different between Model 1, Model 2, and Model without the ZLB.

[9] In Models 1 and 2, the notional interest rate R_t^* does not necessarily coincide with the actual interest rate R_t, even when the latter became positive. This difference is explained by the measurement error we introduce when we estimate the model.

[10] See, for example, Ichiue and Ueno (2006), Krippner (2013), Bauer and Rudebusch (2016), and Ueno (2017).

Because the models are nonlinear, the IRFs differ depending on the state of the economy $(\mu_t^a, Z_t^b, \epsilon_t^r, R_{t-1}^*)$. As an illustration, we express the IRFs conditional on actual states in two historical periods, 1985:1Q and 2010:1Q, as shown in Figure 5. The former is a period in which the nominal interest rate is well over zero, whereas the ZLB constrains the economy in the latter period. Further, because the models are nonlinear, mainly owing to the ZLB, the IRFs are asymmetric to the sign of the shock. Therefore, we show the IRFs to both positive (tightening) and negative (easing) monetary policy shocks.[11] For comparison, we give the same size of the monetary policy shocks, that is, 0.01. The left panels in the figure are the IRFs in 1985:1Q, and the right panels are those in 2010:1Q. The top panels represent the IRFs of the inflation rate π_t, whereas the bottom panels represent those of the nominal interest rate R_t.

For 1985:1Q, the bottom-left panel of Figure 5 shows that the IRFs of the nominal interest rate are similar between Model 1, Model 2, and Model without the ZLB. However, because Model without the ZLB has the smallest inertia (ρ^r) in its policy rule, the nominal interest rate converges to zero most quickly. Therefore, as the top-left panel shows, the IRFs of inflation are the smallest. We also find that the IRFs are symmetric to positive (Pos in the figure) and negative (Neg in the figure) monetary policy shocks.

In 2010:1Q, the IRFs vary significantly, depending on models and on the sign of the monetary policy shock. First, in Model without the ZLB, the IRFs are almost symmetric, because the nominal interest rate can be below zero. The negative monetary policy shock increases the inflation rate and decreases the nominal interest rate, as the right panels show.

Second, we examine the IRFs in Model 1. The negative monetary policy shock has a bigger positive effect on inflation in Model 1 than in Model 2, but the effect is smaller than it is in Model without the ZLB. This result arises because Model 1 involves a strong commitment to future policy. The monetary policy in period t depends on R_{t-1}^*, which can take negative values. The negative monetary policy shock in period t decreases R_t^*, as the bottom-right panel shows, which functions to lower future nominal interest rates. This commitment increases today's inflation rate. The positive monetary policy shock, on the other hand, has a smaller negative effect on inflation in Model 1 compared with in Model 2. Because the monetary policy in period t depends on R_{t-1}^* in Model 1, the positive monetary policy shock in Model 1 induces a smaller increase in the nominal interest rate than that in Model 2 does, as shown in the bottom-right panel.

Third, in Model 2, the negative monetary policy shock has no effect on inflation owing to the ZLB. Even though R_t^* decreases in period 1, this is not sustained after period 2, which weakens the effect of monetary easing. On the other hand, the positive monetary

[11] More precisely, we calculate the generalized impulse response functions following Koop, Pesaran, and Potter (1996). We use the posterior mean estimates in each model and generate the paths of endogenous variables with and without a monetary policy shock. We then subtract the paths with a monetary policy shock from those without a monetary policy shock.

policy shock in Model 2 has a significant negative effect on inflation. This is because the experience of prolonged recessions does not tie the hand of the central bank and, hence, the positive monetary policy shock leads to an immediate increase in the nominal interest rate.

It is impossible to know without estimations which model best reflects reality. We have argued that our estimation supports Model 1, as shown in Table 2. This suggests the commitment effect of monetary policy or the power of forward guidance.

4.4 How Often Does the ZLB Constrain Monetary Policy?

Table 3 shows the probability that the nominal interest rate is equal to zero after $h = 1$, 2, 4, and 8 quarters, given the natural rate of interest. To calculate this, we generate the path of nominal interest rates from $t+1$ conditional on the level of the natural rate of interest r_t^* in period t. We count the number of events using a Monte Carlo simulation. The probability of the ZLB is high, that is, around 60% and 40% for $h = 1$ and 8, respectively, when r_t^* is around 0%. This is significantly higher than the probability of 7.1% reported by Gust et al. (2017) for the United States.[12]

Such a high probability of the ZLB is associated with a high (low) probability of deflation (inflation). We calculate the probability that the inflation rate π_{t+h} reaches 2% (annually), which is the target of the Bank of Japan, or falls below 0% for at least a certain h ($1 \leq h \leq 8$; i.e., within two years). If r_t^* is around 0%, the inflation rate can reach 2% only with a probability of 15%. Although we do not show it here, this probability is even lower for Model 2 and Model without the ZLB, because the estimated π^* is lower. On the other hand, the probability of deflation is considerably high. Even if r_t^* is as high as 3% quarterly, the probability is higher than 75%.

Professionals seem to have similar forecasts for monetary policy and the inflation rate. The Japan Center for Economic Research conducts a survey of forecasts of around 40 professionals every month (ESP forecast). According to the survey conducted between December 25, 2015, and January 5, 2016, which was just before the surprise introduction of the negative interest rate policy, only seven of 35 professionals answered that the next move of monetary policy is tightening. Furthermore, all seven professionals answered that tightening would not start within a year. Then, after one year, according to the survey conducted in December 2016, all 40 professionals answered that the short-term interest rate would remain unchanged, or even decrease, in December 2017. This evidence supports our simulation result that the probability of the ZLB is very high.

With regard to the inflation rate, each forecaster reports the distribution of CPI inflation forecasts. For example, he/she answers the probability that the CPI inflation rate will be between 0 and 0.25%. Taking the mean of each forecaster's forecast distribution for the

[12]Richter and Throckmorton (2016) do not calculate the probability. Instead, they calculate the expected duration of the zero interest rate, with a mode and mean of 1 and 3.2 quarters, respectively.

fiscal year of 2017 from the survey conducted in December 2016, we find that the probability that the inflation rate exceeds 2% is only 0.3%, while the probability that the inflation rate becomes negative is 8.1%. They attach the highest probability of 30% to the inflation rate between 0.5 to 075%. Although the deviation of forecasters' inflation expectations is much smaller, the fact that forecasters expect deflation rather than 2% inflation is consistent with our simulation result.

4.5 Validity of Using the (Un)Constrained Linear Model

Computational costs for the estimation of nonlinear DSGE models with the ZLB are high. One estimation takes almost a week, even in such a simple model as ours. In this regard, Richter and Throckmorton (2016) argue that a constrained linear model performs well, as well as mitigating the computational burden. To examine whether their finding holds for Japan, we estimate the log-linearized DSGE model expressed by equations (9) to (11), but continue to impose the ZLB constraint properly.[13] We call this the constrained linear model. Moreover, we estimate an unconstrained linear model, where we estimate the log-linearized DSGE model by ignoring the ZLB. That is, the unconstrained linear model corresponds to the constrained linear model without the ZLB.

Table 4 shows the estimation results. Although we are able to estimate these two models with reasonable parameter values, we find a large shift for γ^a and π^*, which is important for the steady-state natural rate of interest and the inflation rate. The marginal likelihoods in these models are significantly lower than that in Model 1.

5 Evaluating Japan's Natural Rate of Interest

Next, we turn to our second objective: the natural rate of interest for Japan. After we explain the mathematical expression of the natural rate of interest in our model, we discuss how our nonlinear estimation influences the estimate of the natural rate of interest, how much the natural rate of interest has declined during Japan's lost decades, and why.

5.1 Natural Rate of Interest in the Log-Linearized Model

Our model can be expressed using three key log-linearized equations. Although we do not use them in our nonlinear estimation, they illustrate the role played by the natural rate of interest, r_t^*.[14] They are

[13]The constrained linear model in Richter and Throckmorton (2016) may be different because they explain that they "impose the constraint in the filter but not the solution."

[14]Moreover, they serve as an initial value of equilibrium when we solve the model nonlinearly.

$$\pi_t - \pi^* = \beta e^{(1-\sigma)\gamma^a} E_t\left[\pi_{t+1} - \pi^*\right] + \frac{(\varepsilon - 1)(\omega + \sigma)}{\phi \pi^*}(y_t - y_t^*), \tag{9}$$

$$y_t - y_t^* = E_t\left[y_{t+1} - y_{t+1}^* - \frac{1}{\sigma}\left(\frac{R_t - r^*\pi^*}{r^*\pi^*} - \frac{\pi_{t+1} - \pi^*}{\pi^*} - \frac{r_t^* - r^*}{r^*}\right)\right], \tag{10}$$

and

$$r_t^* = r^*\left[1 + \sigma\rho^a\mu_t^a + (1 - \rho^b)\log Z_t^b)\right]. \tag{11}$$

Here, we define the log-linearized variables of $\{Y_t, Y_t^*\}$ by $\{y_t, y_t^*\}$ around their non-zero trends. Note that the steady-state natural rate of interest r^* is equal to $e^{\sigma\gamma^a}/\beta$.

This suggests that the natural rate of interest r_t^* plays an important role. When r_t^* decreases, both the output gap $y_t - y_t^*$ and the inflation rate π_t decrease, unless the monetary policy is sufficiently strong to offset this. The decrease in r_t^* also causes the nominal interest rate to decrease. Thus, the possibility of reaching the ZLB increases.

The above equations also suggest that the output gap and the inflation rate depend only on the natural rate of interest r_t^* and the actual real interest rate $(\frac{R_t - r^*\pi^*}{r^*\pi^*} - \frac{\pi_{t+1} - \pi^*}{\pi^*})$. We do not need to know μ_t^a and Z_t^b, separately. These shocks influence the output gap and the inflation rate only through a change in r_t^*. Moreover, equation (11) shows that the monetary policy shock ϵ_t^r is irrelevant to the natural rate of interest.

5.2 Developments in the Natural Rate of Interest

In the left panel of Figure 6, we show the time-series path of the natural rate of interest r_t^*. The figure shows a decline in the natural rate of interest. Although it stayed positive until the late 1990s and its steady-state value is 0.46% quarterly, the natural rate of interest often became negative in the 2000s and 2010s. During this period, it often fell to around −0.5% quarterly, and at its worst, to −2%.

The discount factor shock Z_t^b is the most important factor in this result. In the left panel of the figure, we show how much of the model's fit is attributable to individual shocks $(\mu_t^a, Z_t^b, \epsilon_t^r)$. Similar to Gust et al. (2017), we decompose the variable by calculating the model's dynamics, assuming only one of the three shocks is present. Because of the nonlinearity, their sum is not necessarily equal to the natural rate of interest. The right panel of Figure 6 shows the time-series paths of the three types of shocks $(\mu_t^a, Z_t^b, \epsilon_t^r)$. It shows that the discount factor shock Z_t^b explains most of the changes in the natural rate of interest in our estimation periods. The technology shock μ_t^a explains very little of the change in the natural rate of interest, while the monetary policy shock ϵ_t^r does not explain it at all, which is consistent with equation (11).

This result is in line with past studies. Gust et al. (2017) find that the risk premium shock and the marginal efficiency of the investment shock are important in explaining the Great Recession in the United States, while the technology and monetary policy shocks explain little. Sugo and Ueda (2008) estimate the medium-scale DSGE model for Japan, although they estimate a log-linearized model without the ZLB using the sample until 1995, when the ZLB did not constrain the Japanese economy. They also find that the investment shock is the most important. Although our model is far simpler than their models, the discount factor shock in our model is considered to be in the same class as these shocks.

5.3 Comparisons of the Natural Rate of Interest

5.3.1 Different Monetary Policy Specifications

In the previous section, we showed that the nonlinear estimation significantly modifies the implications for monetary policy. Is this true for the natural rate of interest as well? Figure 7 shows how much the estimated natural rate of interest changes, depending on the type of model we estimate. Somewhat surprisingly, the natural rate of interest in Model 2 and in Model without the ZLB are not significantly different from that in Model 1, even though Model 2 and Model without the ZLB yield biased estimates, as shown in Table 2. In particular, during the period of the ZLB since 2000, the natural rate of interest is almost identical, although it deviates in the 1980s and 1990s. This result is in sharp contrast to that of Hirose and Sunakawa (2017). Although this difference may come from a difference in data (i.e., they estimate a model for the United States, not for Japan), it is important to note a difference in methodology. Instead of estimating a DSGE model with the ZLB, they estimate the model without the ZLB for the periods before the ZLB constrains the economy and then evaluate the natural rate of interest using the estimated parameters for the extended periods. They then find that the natural rate of interest is substantially higher when considering the ZLB than when neglecting the ZLB, particularly during the ZLB period.

To understand why there is so little variation in the estimated natural rate of interest in our study, we identify three types of differences between Model 1 and Model without the ZLB. They are (1) the presence of the ZLB, (2) estimated parameters, and (3) estimated shocks. More specifically, we simulate the natural rate of interest in Model without the ZLB by changing one of the three differences: (1) using estimated parameters and shocks in Model without the ZLB, but now explicitly taking into account the ZLB, (2) using the estimated parameters in Model 1, and (3) using the estimated shocks in Model 1. Here, (2) is analogous to Hirose and Sunakawa (2017).

Figure 8 shows the paths of the counterfactual natural rate of interest. As in the case of type (1), we show that the presence of the ZLB does not influence the natural rate of interest, per se. Because the natural rate of interest rests on the flexible-price economy, by definition, the ZLB does not matter in its movements, per se. For types (2) and (3), the figure suggests that the parameter difference has a similar size, but opposite effect on the natural rate of interest as that of the shock difference. Type (2) increases the natural rate of interest, as in Hirose and Sunakawa (2017). This stems from the fact that the estimate of the steady-state natural rate of interest r^* in Model 1 is higher than that in Model without the ZLB as shown in Table 2. However, type (3) decreases the natural rate of interest by the same size, because the estimated shocks of (μ_t^a, Z_t^b) are lower for Model 1. This result suggests that we should estimate the parameters and shocks simultaneously.

5.3.2 Laubach-Williams (2003) and Hodrick–Prescott Filter

Next, we compare the natural rate of interest based on our model with that based on Laubach and Williams (2003) and that based on the Hodrick–Prescott (HP) filter. Laubach and Williams (2003) and Holston, Laubach, and Williams (2017) estimate the backward-looking IS and Phillips curves jointly and calculate the natural rate of interest using the Kalman filter, where they calculate the *ex ante* real interest rate by estimating the one-year ahead inflation expectation from a univariate AR(3) model. In their model, the natural rate of interest is given by

$$r_t^* = g_t + z_t, \tag{12}$$

where g_t and z_t capture the trend growth rate of the natural output and other determinants such as demand disturbances, respectively. Therefore, g_t and z_t in their model correspond to $\log(A_t/A_{t-1}) = \mu_t^a + \gamma^a$ and $\log Z_t^b$ in our model, respectively. However, there is one important difference. Laubach and Williams (2003) and Holston, Laubach, and Williams (2017) assume that both g_t and z_t obey an I(1) process (i.e., the natural output is I(2)), while both $\log(A_t/A_{t-1})$ and $\log Z_t^b$ in our model obey an I(0) process.

We apply their approach to the Japanese data and report the one-sided (filtered) estimate of the natural rate of interest. For the HP filter, we set the smoothing parameter λ to 1,600 and smooth the same *ex ante* real interest rate we used to calculate the natural rate of interest based on Laubach and Williams (2003).

Figure 9 shows that the one-sided estimate of the natural rate of interest based on Laubach and Williams (2003) moves very closely to that based on our model. Although Laubach and Williams's (2003) approach does not take into account the ZLB, the movements of the natural rate of interest are similar when the nominal interest rate is effectively at the ZLB. This result is consistent with our previous finding that the nonlinear estimation barely changes the estimate of the natural rate of interest, as shown in Figures 7 and 8. Furthermore, we confirm that most of the fluctuations of the natural rate of interest based on Laubach and Williams (2003) are caused by z_t, although we do not show this here. This result is again consistent with ours. The fluctuations of the natural rate of interest based on the HP filter are far smoother, although their means do not change significantly.

5.3.3 Use of the Output Gap Data

Finally, we check the robustness of our estimation using an alternative measure of output, that is, the output gap. We estimate the same model using either the output gap instead of the growth of real GDP or using both the output gap and the growth of real GDP. Table 5 shows that the parameter estimates are much the same. However, as Figure 10 shows, the path of the natural rate of interest comes to differ quantitatively. In particular, when we use the output gap instead of the growth of real GDP, the natural rate of interest becomes more volatile. Nevertheless, qualitatively, the three lines are very similar.

6 Concluding Remarks

In this study, we estimated a nonlinear DSGE model with the ZLB using a Bayesian technique. There are several potential avenues for future research. The first would be to estimate a richer DSGE model, embedding capital, wage stickiness, financial frictions, and so on. We are aware that the intrinsic persistence in our model is low, which makes the economy return to the steady state rather quickly. This could be one reason why we succeeded in estimating the nonlinear DSGE model with the ZLB, even though the duration of the ZLB is relatively long in Japan. However, embedding these features poses a computational challenge owing to the curse of dimensionality. Moreover, we suspect that it becomes more difficult to find a determinate equilibrium for a set of parameters, because such stickiness lengthens the duration of the ZLB further, and makes it more likely that the equilibrium will be indeterminate, as argued by Aruoba, Cuba-Borda, and Schorfheide (2017).

This point leads to the second avenue for our future research: estimating a regime-switching model. In particualr, we consider three types of regime shifts. As in Aruoba, Cuba-Borda, and Schorfheide (2017), the equilibrium may fluctuate between a normal determinate equilibrium and a deflationary indeterminate equilibrium. Alternatively, the equilibrium may fluctuate between the regime of active monetary policy and passive fiscal policy and the regime of passive monetary policy and active fiscal policy. Another option is that there could be discontinuous changes in some structural parameters, such as the steady-state growth rate of technology (γ^a) and the inflation target (π^*). Most importantly, the existence of a kink has been often pointed out for the GDP growth rate when a big adverse shock hits the economy (around 1991 for Japan and 2008 for the United States), which may call for a regime-switching model for γ^a.

Third, our method can be applied to other types of models, where nonlinearity plays an important role. Examples include currency and financial crises, where crises occur as a tail-risk event and have significant impacts on the economy.

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A Appendix

A.1 Model Solution

We derive the rational expectation equilibrium of our model using the *time iteration method with linear interpolation* (TL) method. The model's equilibrium conditions are written as a vector-valued function, $f(\cdot)$, containing the minimum state vector $(\mathbf{v}_t, \mathbf{w}_t)$, say

$$E[f(\mathbf{v}_{t+1}, \mathbf{w}_{t+1}, \mathbf{v}_t, \mathbf{w}_t)|\Omega_t] = 0,$$

where $\mathbf{v}$ is a vector of exogenous variables, $\mathbf{w}$ is a vector of endogenous variables, and Ω_t is an information set of agents. In our study, there are three models, $\mathbf{M} = \{$Model 1, Model 2, Model w/o ZLB$\}$, and $\mathbf{v} = \left\{\epsilon_t^a,\, \epsilon_t^b,\, \epsilon_t^r\right\}$, $\mathbf{w} = \left\{y_t, c_t, \pi_t, y_t^*, R_t, R_t^*, r_t^*, \mu_t^a, z_t^b\right\}$ and $\mathbf{z} = \left\{R_{t-1}^*,\, \mu_t^a,\, z_t^b,\, \epsilon_t^r\right\}$, where $y_t \equiv Y_t/A_t$, $c_t \equiv C_t/A_t$, and $y_t^* \equiv Y_t^*/A_t$.

Given function $f(\cdot)$, we can obtain a model's decision rules (or policy functions), $\Phi(\centerdot)$, as a function of the state vector. The TL locally approximates the time-invariant policy function at each node in the state space, $\mathbf{z_t}$, that is,

$$\Phi(\mathbf{z}_t) \simeq \hat{\Phi}(\mathbf{z}_t).$$

We choose to iterate on $\Phi(\mathbf{z}_t)$ for $\{\, y_t,\, y_t^*,\, \pi_t \,\}$ and solve for the rational expectations equilibrium by substituting $\Phi(\mathbf{z}_t)$ into the future variables of the function $f(\cdot)$. We discretize nine grid points on each of the continuous state variables, $\left\{R_{t-1}^*,\, \mu_t^a,\, z_t^b\right\}$, and five grid points on the exogenous shock of ϵ_t^r, which implies $3,645 (= 9 * 9 * 9 * 5)$ nodes in total.

The policy function iteration algorithm takes the following steps. Let $i \in \{0, \cdots, I\}$ denote the iterations of the algorithm and $n \in \{1, \cdots, N\}$ denote the nodes of the policy function, $\Phi(\mathbf{z}_t)$.

1. For $i = 0$, we make an initial conjecture of the policy function, $\Phi^0(\mathbf{z}_t)$, from the log-linearized model without the ZLB. To do so, we use Sims' (2002) `gensys` algorithm.

2. For iteration $i \in \{1, \cdots, I\}$ and node $n \in \{1, \cdots, N\}$, we execute the following procedures:

   (a) Solve for endogenous variables $\left\{c_t, R_t, R_t^*, r^*, \mu_t^a, z_t^b\right\}$, given $y(\mathbf{z}_{t-1})$, $y^*(\mathbf{z}_{t-1})$, $\pi(\mathbf{z}_{t-1})$ under the ZLB.

   (b) Approximate the future variables $\{\, \mathrm{E}(\pi_{t+1}),\, \mathrm{E}(y_{t+1}),\, \mathrm{E}(y_{t+1}^*)\}$ using a piecewise linear interpolation of the policy function $\Phi^{i-1}(z_t)$. Then, substitute the future variables into $E[f(\cdot)|\Omega_t]$, conditional on the exogenous variables in the next period, $\mathbf{v}_{t+1}$, equal to zero.[15]

[15] In this respect, our estimated model is not purely stochastic. We tried to estimate the model by

(c) Use the nonlinear solver, Sims' `csolve`, to find the policy function, $\Phi^i(\mathbf{z_t})$, which minimizes the errors in intertemporal equations, that is, $E[f(\cdot)|\Omega_t] = 0$.

3. Define $\texttt{maxdist} = \texttt{max}(\,|y_n^i - y_n^{i-1}|,\ |y_n^{*,i} - y_n^{*,i-1}|,\ |\pi_n^i - \pi_n^{i-1}|)$. Repeat step 2 until the policy function converges, say, to $\texttt{maxdist} < 10^{-4}$, for all nodes, n.

A.2 Estimation

To obtain draws from the posterior distribution of parameters, θ, of a nonlinear DSGE model, we use the *Sequential Monte Carlo Squared* (SMC^2) sampler, combined with the *particle filter*, instead of popular methods such as the MCMC sampler. Because MCMC samplers cannot be parallelized when generating the draws, they take quite a long time. In contrast, the SMC^2 method and particle filter can be used easily and, in addition, may calculate a more accurate approximation of the posterior distribution than that of the MCMC samplers. We explain the algorithms of the SMC^2 and particle filter following Herbst and Schorfheide (2015) and Fernandez-Villaverde et al. (2016).

A.2.1 Algorithm of the Sequential Monte Carlo Squared

Suppose ϕ_n, for $n = 0, \cdots, N_\phi$, is a sequence that slowly increases from zero to one. We define a sequence of bridge distributions, $\{\pi_n(\theta)\}_{n=0}^{N_\phi}$, that converge to the target posterior distribution for $n = N_\phi$ and $\phi_n = 1$, as

$$\pi_n(\theta) = \frac{[p(Y|\theta)]^{\phi_n} p(\theta)}{\int [p(Y|\theta)]^{\phi_n} p(\theta) d\theta}, \quad \text{for } n = 0, \cdots, N_\phi,\ \ \phi_n \uparrow 1,$$

where $p(\theta)$ and $p(Y|\theta)$ are the prior density and likelihood function, respectively. We adopt the likelihood tempering approach that generates the bridge distributions, $\{\pi_n(\theta)\}_{n=0}^{N_\phi}$, by taking power transformation of $p(Y|\theta)$ with the parameter, ϕ_n, that is, $[p(Y|\theta)]^{\phi_n}$.

The SMC^2 with the likelihood tempering takes the following steps. Let $i \in \{1, \cdots, N_\theta\}$ denote the particles of the parameter sets, θ^i, and $n \in \{0, \cdots, N_\phi\}$ denote the stage of the algorithm. Herbst and Schorfheide (2015) recommend a convex tempering schedule in the form of $\phi_n = (n/N_\phi)^\lambda$ with $\lambda = 2$ for a small-scale DSGE model.

1. Initialization

   (a) Set the initial stage as $n = 0$, and draw the initial particles of parameters, θ_0^i, from a prior distribution $p(\theta)$.

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considering that the exogenous variables in the next period, $\mathbf{v}_{t+1}$ obey a normal distribution, but could not obtain reasonable results. We suspect this is because there is no equilibrium or that the equilibrium becomes indeterminate or unstable in an economy such as Japan, where the ZLB has constrained the monetary policy for a long time. See Hills, Nakata, and Schmidt (2016).

(b) Set the weight of each particle of the initial stage as $W_0^i = 1$, for $i = 1, \cdots, N_\theta$.

Then, for stage $n \in \{1, \cdots, N_\phi\}$ and particle $i \in \{1, \cdots, N_\theta\}$, we repeat Steps 2 to 4 below.

2. Correction. Calculate the normalized weight, $\tilde{W}_n^i$, for each particle as

$$\tilde{W}_n^i = \frac{\tilde{w}_n^i W_{n-1}^i}{\frac{1}{N}\sum_{i=1}^{N} \tilde{w}_n^i W_{n-1}^i} \text{ for } i = 1, \cdots, N_\theta,$$

where $\tilde{w}_t^i$ is an incremental weight derived from

$$\tilde{w}_n^i = [p(Y|\theta_{n-1}^i)]^{\phi_n - \phi_{n-1}},$$

and the likelihood, $\hat{p}(Y|\theta)$, is approximated from the particle filter, as explained in the next subsection.

Note that the correction step is a classic importance sampling step, in which particle weights are updated to reflect the stage n distribution, $\pi_n(\theta)$. Because this step does not change the particle value, we can skip this step only by calculating the power transformation of $p(Y|\theta)$ with the parameter, ϕ_n.

3. Selection (Resampling).

(a) Calculate an effective particle sample size, $\widehat{ESS_n}$, which is defined as

$$\widehat{ESS_n} = N_\theta / \left(\frac{1}{N_\theta} \sum_{i=1}^{N_\theta} (\tilde{W}_n^i)^2 \right).$$

(b) If $\widehat{ESS_n} < N_\theta/2$, then resample the particles, $\{\hat{\theta}_n\}_{i=1}^{N_\theta}$, via multinomial resampling and set $W_n^i = 1$.

(c) Otherwise, let $\hat{\theta}_n^i = \theta_{n-1}^i$ and $W_n^i = \tilde{W}_n^i$.

4. Mutation. Propagate the particles $\{\hat{\theta}_n^i, W_n^i\}$ via the random walk MH algorithm with the proposal density,

$$\vartheta|\hat{\theta_n}^i \sim N\left(\hat{\theta_n}^i, c_n^2 \Sigma\left(\hat{\theta_n}\right)\right),$$

where $N(\cdot)$ is the nominal distribution and $\Sigma\left(\hat{\theta_n}\right)$ denotes the covariance matrix of parameter $\hat{\theta_n}$ for all particles $i \in \{1, \cdots, N_\theta\}$ at the n-th stage. In order to keep the acceptance rate around 25%, we set a scaling factor c_n for $n > 2$ as

$$c_n = c_{n-1} f(A_{n-1}),$$

where A_n represents the acceptance rate in the mutation step at the n-th stage and the function $f(x)$ is given by

$$f(x) = 0.95 + 0.10 \frac{e^{16(x-0.25)}}{1 + e^{16(x-0.25)}}.$$

5. For the final stage of $n = N_\phi$, calculate the final importance sampling approximation of posterior estimator, $E_\pi[h(\theta)]$, as

$$h_{N_\phi,N_\theta} = \sum_{i=1}^{N_\theta} h(\theta^i_{N_\phi}) W^i_{N_\phi}.$$

We note that, in the final stage, the approximated marginal likelihood of the model is also obtained as a by-product. It can be shown that

$$P_{SMC}(Y) = \prod_{n=1}^{N_\phi} \left(\frac{1}{N_\theta} \sum_{i=1}^{N_\theta} \tilde{w}^i_n W^i_{n-1} \right)$$

converges almost surely to $p(Y)$ as the number of particles $N_\theta \to \infty$.

A.2.2 Algorithm of the Particle Filter

Suppose that a state-space representation for the nonlinear DSGE model consists of

$$y^{obs}_t = \Psi(s_t,\ \theta) + u_t, \quad u_t \sim N(0,\, \Sigma_u),$$

$$s_t = \Phi(s_{t-1},\ \varepsilon_t,\ \theta), \quad \varepsilon_t \sim N(0,\, \Sigma_\varepsilon),$$

where y^{obs}_t and s_t denote the observable and state variables, respectively. In our study, we set $y^{obs}_t = \{\ln(y_t/y_{t-1}),\, \pi_t,\, R_t\}$ and $\mathbf{s_t} = \left\{ y_t,\, c_t,\, \pi_t,\, y^*_t,\, R_t,\, R^*_t,\, r^*_t,\, \mu^a_t,\, z^b_t \right\}$. A measurement error vector, u_t, and an exogenous shock vector, $\varepsilon_t = \left\{\epsilon^a_t,\, \epsilon^b_t,\, \epsilon^r_t\right\}$, follow a normal distribution with covariance matrices, Σ_u and Σ_ε, respectively. The nonlinear policy functions, $\Phi(s_t,\ \varepsilon_t,\ \theta)$, are derived from Appendix A.1, while the function $\Psi(s_t, \theta)$ represents the linkage between y^{obs}_t and s_t.

The particle filter algorithm is as follows. Let $j \in \{0, \cdots, N_S\}$ denote the particles of the state variables and exogenous shocks.

1. For period $t = 0$, draw the N_S initial particles of the state variables at period 0, say $s^j_{0|0}$, from the distribution around the steady state derived from the policy functions, $p(s_0 \mid \Sigma_\varepsilon,\, \theta)$.

2. For period $t \in \{1, \cdots, T\}$ and particle $j \in \{1, \cdots, N_S\}$, take the following three steps.

   (a) Step of forecasting the state variables: $s^j_{t|t-1}$. Generate N_S particles of the shock vector ε^j_t from $N(0,\, \Sigma_\varepsilon)$. Using the nonlinear policy function, we obtain N_S particles of forecasts of the state variables corresponding to the shocks generated in the above:

   $$s^j_{t|t-1} = \Phi(s^j_{t-1|t-1}, \varepsilon^j_t | \theta).$$

(b) Step of forecasting the observable variables. Calculate the approximated predictive density of y_t^{obs} given by

$$p(y_t^{obs}|Y_{1:t-1}^{obs},\, \theta) \doteqdot \frac{1}{N_S}\sum_{j=1}^{N_S} w_t^j,$$

where w_t^j is the normal predictive density of the particle j measured from $\Psi(s_{t|t-1}^j, \theta)$ and the covariance matrix of the measurement error Σ_u in period t, say,

$$w_t^j = (2\pi)^{-n/2}|\Sigma_u|^{-1/2}\mathrm{exp}\left\{ -\frac{1}{2}\,(y_t^{obs} - \Psi(s_{t|t-1}^j, \theta))'\,\Sigma_u^{-1}\,(y_t^{obs} - \Psi(s_{t|t-1}^j, \theta)) \right\},$$

where n is the dimension of y_t.

(c) Step of updating the state variables: $s_{t|t}^j$. Resample N_S particles of the state variables from a multinomial distribution. That is,

$$s_{t|t}^j = \text{resample out of } (\mathrm{s}_{\mathrm{t|t-1}}^1, \cdots \mathrm{s}_{\mathrm{t|t-1}}^{\mathrm{j}}, \cdots \mathrm{s}_{\mathrm{t|t-1}}^{\mathrm{N_s}}) \text{ with probabilty } (w_t^j/\Sigma w_t^j).$$

3. For the final period of $t = T$, collect all predictive densities of y_t from period 1 to T, calculated above. Using these, the log likelihood of the model is approximated as

$$\ln p(Y_{1:t}^{obs}|\theta) \doteqdot \sum_{t=1}^{T} \ln\left(\frac{1}{N_S}\sum_{j=1}^{N_S} w_t^j\right).$$

Table 1: Prior Distribution

Parameter	Mean	S.D.	Shape	Parameter	Mean	S.D.	Shape
σ	1.5	0.3	Normal	ρ^r	0.5	0.2	Beta
γ^a	0	0.5	Normal	ψ_π	1.5	0.15	Normal
ω	3	0.5	Normal	ψ_y	0.125	0.025	Normal
κ	0.05	0.006	Normal	ω	3	0.5	Normal
π^*	0	0.5	Normal	ρ^a	0.5	0.2	Beta
				$\sigma^a, \sigma^b, \sigma^r$	$\sqrt{0.02}$	5 (d.f.)	Inv Gamma

Table 2: Posterior Distribution and Marginal Likelihood

	Model 1		Model 2		Model w/o ZLB	
Parameter	Mean	(95% low, high)	Mean	(95% low, high)	Mean	(95% low, high)
σ	1.400	(1.289, 1.55)	1.534	(1.472, 1.584)	1.037	(1.004, 1.083)
γ^a	-0.028	(-0.119, 0.05)	0.123	(0.068, 0.172)	-0.419	(-0.539, -0.31)
ω	2.477	(2.298, 2.674)	3.163	(2.908, 3.413)	3.188	(3.071, 3.287)
κ	0.055	(0.05, 0.062)	0.053	(0.05, 0.055)	0.047	(0.045, 0.049)
π^*	0.360	(-0.117, 0.619)	0.050	(-0.148, 0.274)	-0.447	(-0.728, -0.246)
r^*	0.464	(0.343, 0.574)	0.691	(0.607, 0.764)	0.067	(-0.055, 0.181)
ρ^r	0.521	(0.475, 0.611)	0.685	(0.644, 0.721)	0.214	(0.182, 0.24)
ψ_π	1.689	(1.627, 1.745)	1.776	(1.739, 1.811)	1.509	(1.453, 1.553)
ψ_y	0.105	(0.091, 0.123)	0.113	(0.098, 0.125)	0.133	(0.127, 0.141)
ρ^a	0.254	(0.129, 0.446)	0.201	(0.096, 0.292)	0.122	(0.093, 0.147)
ρ^b	0.750	(0.693, 0.802)	0.754	(0.728, 0.776)	0.740	(0.689, 0.788)
σ^a	1.175	(0.906, 1.367)	1.320	(1.146, 1.524)	1.773	(1.675, 1.859)
σ^b	1.797	(1.435, 2.318)	2.229	(2.018, 2.442)	1.354	(1.191, 1.558)
σ^r	1.439	(1.14, 1.673)	1.173	(1.022, 1.349)	0.921	(0.809, 1.012)
Likelihood	-261.753		-275.9136		-364.815	

Table 3: Probability of the ZLB, 2% Inflation (Annually), and Deflation

Natural rate (r_t^*)	Prob ($R_{t+h} = 0\%$ | r_t^*)				Prob ($\pi_{t+h} > 0.5\% \| r_t^*$)	Prob ($\pi_{t+h} < 0\% \| r_t^*$)
	for $h = 1$	for $h = 2$	for $h = 4$	for $h = 8$	$^{\exists}h$ for $1 \leq h \leq 8$	
$r_t^* \geq 3\%$	7.00%	14.35%	23.35%	31.65%	60.20%	76.60%
$2\% \leq r_t^* < 3\%$	12.64%	19.59%	26.37%	33.80%	47.16%	81.28%
$1\% \leq r_t^* < 2\%$	25.97%	28.62%	32.60%	35.61%	28.44%	88.10%
$0\% \leq r_t^* < 1\%$	52.40%	44.56%	39.94%	38.17%	15.36%	94.11%
$-1\% \leq r_t^* < 0\%$	61.86%	51.74%	43.58%	39.43%	10.40%	97.48%
$r_t^* < -1\%$	69.60%	58.83%	51.60%	41.63%	7.03%	99.70%
$r_t^* = r^*$	35.12%	36.79%	37.52%	37.29%	15.16%	94.26%

Table 4: Estimation Results of the (Un)Constrained Linear Model

	Model 1 (benchmark)		Constrained linear (Model 1)		Unconstrained linear (Model w/o ZLB)	
Parameter	Mean	(95% low, high)	Mean	(95% low, high)	Mean	(95% low, high)
σ	1.400	(1.289, 1.55)	1.287	(1.237, 1.387)	1.152	(1.039, 1.249)
γ^a	-0.028	(-0.119, 0.05)	0.140	(0.031, 0.22)	0.544	(0.44, 0.633)
ω	2.477	(2.298, 2.674)	2.702	(2.569, 2.936)	2.260	(2.125, 2.377)
κ	0.055	(0.05, 0.062)	0.050	(0.048, 0.052)	0.047	(0.046, 0.048)
π^*	0.360	(-0.117, 0.619)	-0.138	(-0.316, -0.03)	0.844	(0.69, 1.006)
r^*	0.464	(0.343, 0.574)	0.679	(0.541, 0.792)	1.126	(1.037, 1.207)
ρ^r	0.521	(0.475, 0.611)	0.386	(0.354, 0.406)	0.414	(0.377, 0.449)
ψ_π	1.689	(1.627, 1.745)	1.426	(1.379, 1.476)	1.534	(1.503, 1.563)
ψ_y	0.105	(0.091, 0.123)	0.127	(0.123, 0.13)	0.124	(0.116, 0.132)
ρ^a	0.254	(0.129, 0.446)	0.157	(0.115, 0.211)	0.206	(0.183, 0.239)
ρ^b	0.750	(0.693, 0.802)	0.659	(0.647, 0.682)	0.698	(0.666, 0.727)
σ^a	1.175	(0.906, 1.367)	1.285	(1.131, 1.412)	1.649	(1.534, 1.773)
σ^b	1.797	(1.435, 2.318)	1.483	(1.418, 1.565)	1.867	(1.617, 2.066)
σ^r	1.439	(1.14, 1.673)	1.182	(1.023, 1.41)	1.065	(0.979, 1.154)
Likelihood	-261.75		-279.29		-395.65	
Post prob	1.000		0.000		0.000	

Table 5: Estimation Results when Using the Output Gap Data

Parameter	Growth Data (benchmark) Mean	(95% low, high)	Gap instead of growth Mean	(95% low, high)	Gap and growth Mean	(95% low, high)
σ	1.400	(1.289, 1.55)	1.596	(1.547, 1.686)	1.513	(1.468, 1.586)
γ^a	-0.028	(-0.119, 0.05)	-0.056	(-0.088, -0.003)	-0.077	(-0.132, -0.041)
ω	2.477	(2.298, 2.674)	3.030	(2.81, 3.137)	2.743	(2.585, 2.841)
κ	0.055	(0.05, 0.062)	0.044	(0.042, 0.046)	0.055	(0.054, 0.056)
π^*	0.360	(-0.117, 0.619)	0.154	(0.078, 0.186)	0.648	(0.549, 0.714)
r^*	0.464	(0.343, 0.574)	0.414	(0.364, 0.497)	0.385	(0.292, 0.441)
ρ^r	0.521	(0.475, 0.611)	0.664	(0.645, 0.675)	0.597	(0.575, 0.631)
ψ_π	1.689	(1.627, 1.745)	1.468	(1.458, 1.477)	1.431	(1.407, 1.472)
ψ_y	0.105	(0.091, 0.123)	0.113	(0.11, 0.116)	0.101	(0.098, 0.103)
ρ^a	0.254	(0.129, 0.446)	0.202	(0.153, 0.233)	0.222	(0.157, 0.262)
ρ^b	0.750	(0.693, 0.802)	0.680	(0.673, 0.687)	0.801	(0.794, 0.81)
σ^a	1.175	(0.906, 1.367)	3.473	(2.988, 4.367)	1.412	(1.202, 1.766)
σ^b	1.797	(1.435, 2.318)	2.285	(2.225, 2.383)	1.946	(1.623, 2.133)
σ^r	1.439	(1.14, 1.673)	0.823	(0.773, 0.906)	1.473	(1.424, 1.535)

Figure 1: Data

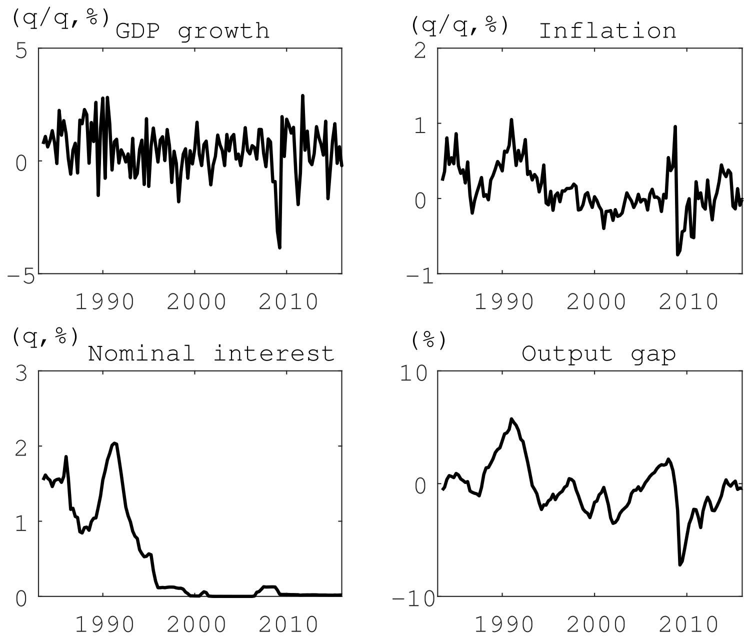
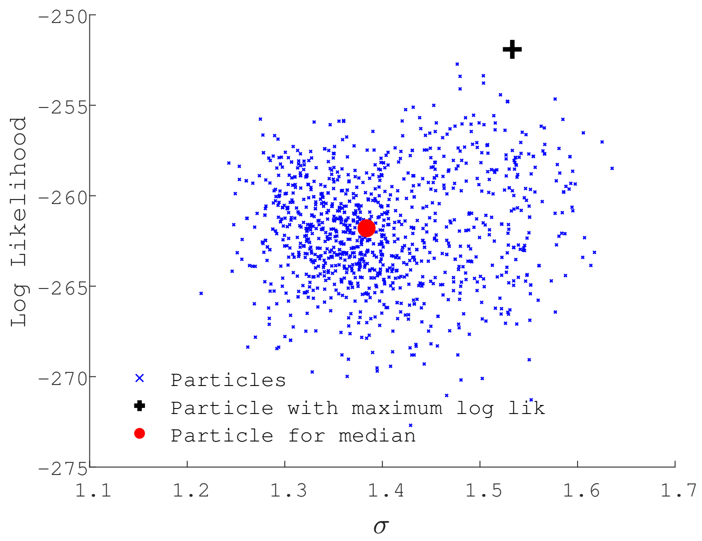


Figure 2: Particles for Parameter σ and their Likelihood



Note: Each dot represents a particle for the value of parameter σ and its posterior likelihood. The big filled circle in red indicates the median, whereas the big plus at the top indicates the maximum of the likelihoods in all the dots.

Figure 3: Notional Nominal Interest Rate R_t^*

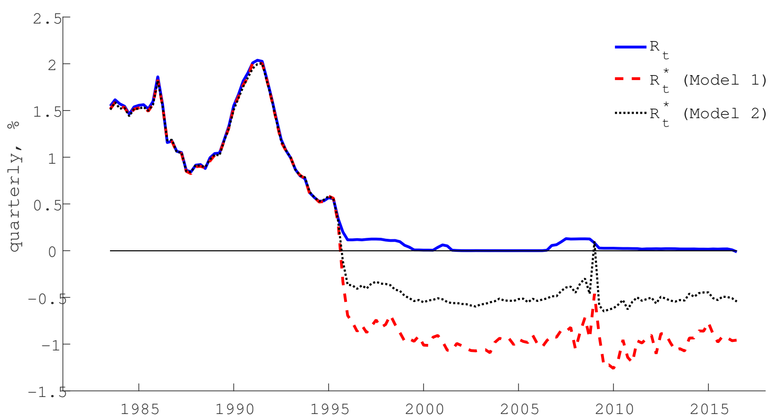


Figure 4: Comparison with the Notional Nominal Interest Rate and the Shadow Rate

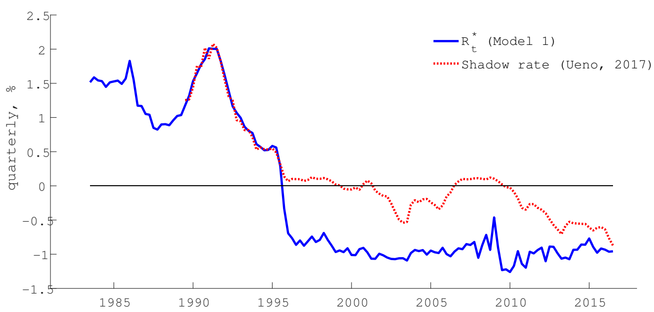
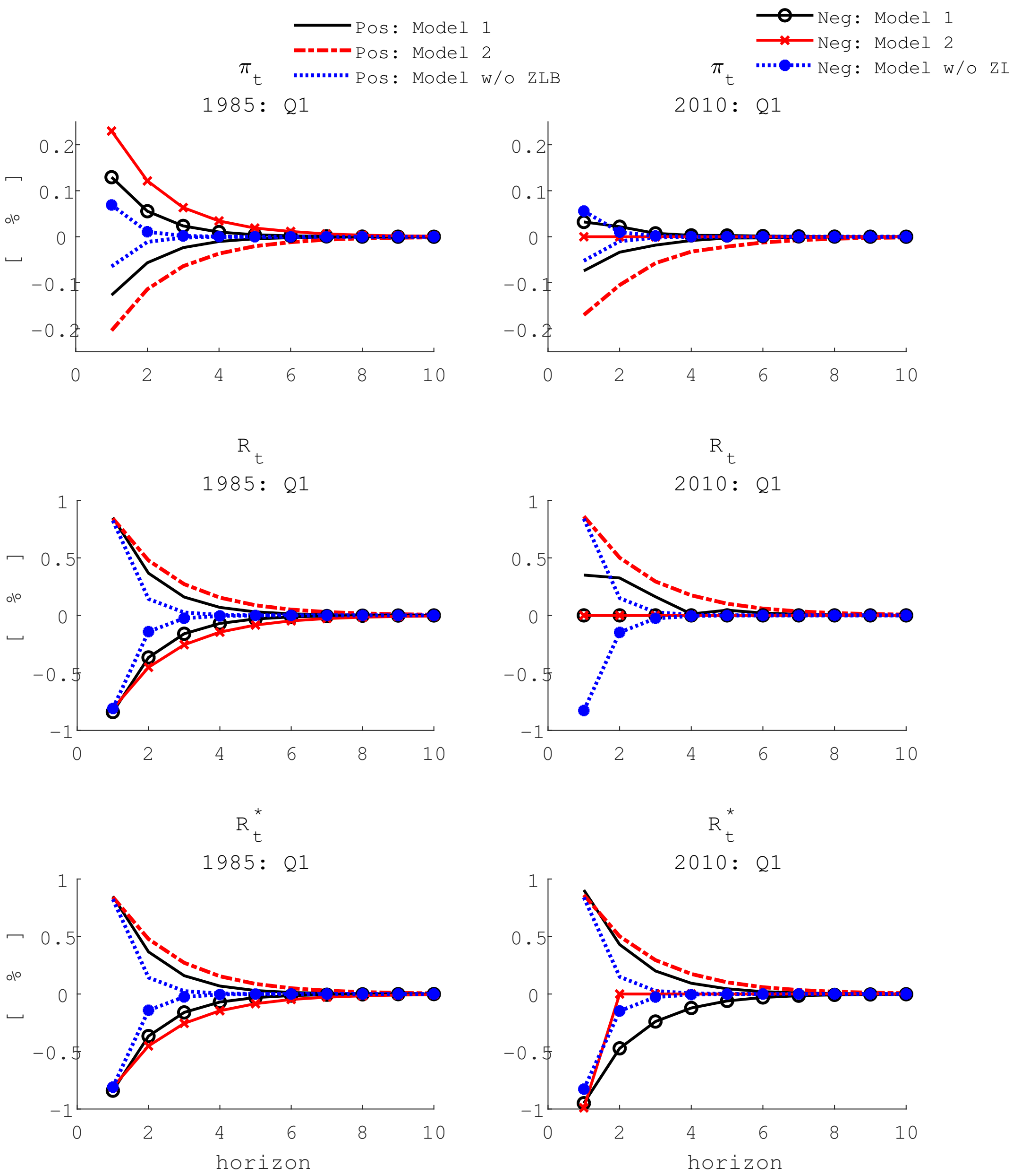


Figure 5: Impulse Responses to a Monetary Policy Shock



Note: Pos and Neg represent positive (tightening) and negative (easing) monetary policy shocks, respectively.

Figure 6: Natural Rate of Interest and the Contribution of the Estimated Shocks

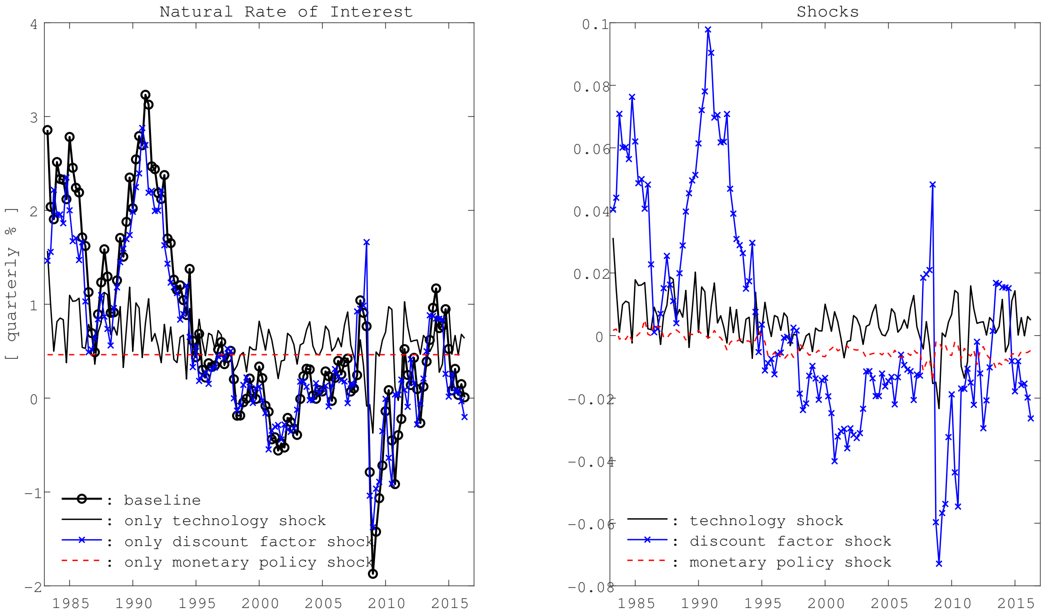


Figure 7: Natural Rate of Interest: Model Comparison (1)

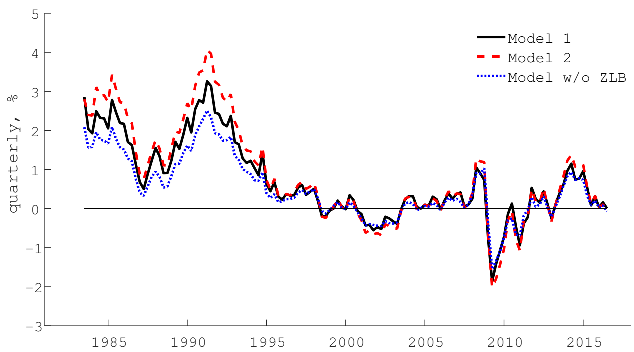
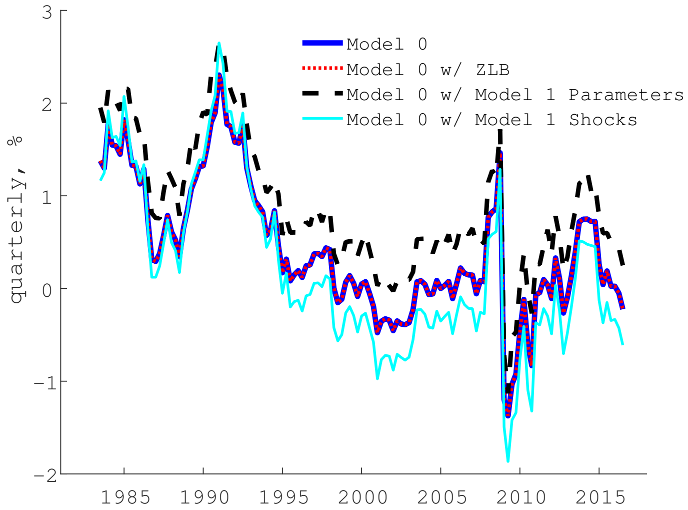


Figure 8: Natural Rate of Interest: Counterfactual Simulation



Note: "Model 0" represents the natural rate of interest based on the Model without the ZLB. (1) "Model 0 w/ ZLB," (2) "Model 0 w/ Model 1 Parameters," and (3) "Model 0 w/ Model 1 Shocks" represent the simulated natural rate of interest using (1) estimated parameters and shocks in the Model without the ZLB, but now explicitly taking into account the ZLB, (2) using estimated parameters in Model 1, and (3) using estimated shocks in Model 1, respectively.

Figure 9: Natural Rate of Interest: Model Comparison (2)

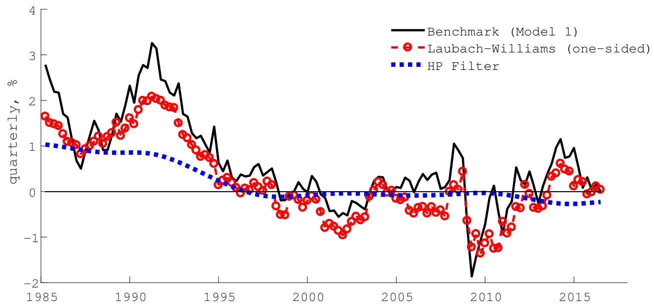


Figure 10: Natural Rate of Interest When Using the Output Gap Data